Initial data sets, conformal geometry and the topology of physical space in general relativity

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Abstract. *We discuss the interplay between the energetic content of physical space and the topology of the underlying three-manifold. Within the context of the conformal approach to the initial value problem we examine both the case of asymptotically euclidean data given on a complete, non-compact three-manifold, and the case of data assigned on a closed three-manifold. In the former case we provide a description, in the full theory, of the topological changes induced by large concentration of gravitational radiation, and of the formation of apparent horizons for time-symmetric data. In the closed case, we show that the presence of enough matter and radiation necessarily implies that the topology of the underlying three-manifold is (up to identifications) the three-sphere topology, or the* $(S^1 \times S^2)$*-wormhole topology, or that of a connected sum of a denumerable number of such manifolds. We also show that such topologies leave, as for as the field equations are concerned, more room to possible gravitational initial data sets.*

1. INTRODUCTION

In general relativity the notion of three-dimensional physical space is just an observer-dependent aspect of the given spacetime geometry. Nonetheless, its

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geometric characterization acquires a particular relevance whenever we wish to consider the gravitational field as a dynamical system and general relativity as an hamiltonian theory describing the spacetime structure as the time evolution of a space-geometry. In this setting, looking at the physical space at a particular instant of its evolution corresponds to choosing a spacelike embedding of a three-dimensional manifold S in the given spacetime. As is known, the geometry of such embedding cannot be arbitrarily given, it is rather a dynamical variable the instantaneous state and future evolution of which is governed by Einstein's field equations. In particular, four among such equations, reduce to constraints that describe the feed-back between the geometry of the physical space and its energetic content. This feed-back is not purely local in the sense that the equations describing the above constraints are elliptic and a slight alteration in the instantaneous distribution of the sources affects the whole geometry of the physical space. The constraints are also non linear, so that the geometry of physical space is affected not only by external sources, but also by the self--interaction of the geometrical field variables representing the gravitational field. In view of this fact, we are naturally led to inquire to what extent the global properties of the physical space, and in particular the topology of the underlying three-manifold S modelling it, are affected by the presence of matter and gravitational radiation. This is an old problem the interest in which has been kept alive not only by the necessity of a deeper understanding of the structure of Einstein's equations, but also by the hope that it may shed some light on the nature of quantum fluctuations in the topology of S (if indeed they occur at all [1]). A thorough and simple answer to this problem is difficult to provide. Technical reasons for such difficulties are connected to the fact that the above mentioned constraints relate the instantaneous distribution of matter and radiation to the scalar curvature of the hypersurface S. And, as is known, the scalar curvature of a manifold gives (in any dimension greater than two) very little information on the behaviour of the geodesic field and hence on the topology. It is also clear from the onset that the field equations do not easily accomodate the topology of the physical space on a dynamical footing. For, it is known that any spacetime manifold without causal anomalies (such as closed timelike curves) becomes singular wherever its spatial sections undergo a change in topology. In particular, if we assume that the spacetime M resulting from the evolution of the data associated with S is globally hyperbolic, then necessarily $M \simeq S \times I$, (I being a suitable subset of $\mathbb{R}$). Since the dynamics of the field is ultimately determined by the constraints (modulo boundary terms [2]) it follows from the above remarks that the constraints are a manifestation of a mechanism that works for preserving the topology of the physical space [1].

An examination of the nature of the constraints is also the key for understanding the interplay between the topology of S and its energetic content. This fact has been clear since the pioneering work of Brill [3] on the solvability of the constraints for vacuum time-symmetric, axi-symmetric, asymptotically Euclidean initial data sets. That analysis (in particular through the work of J.A. Wheeler [4]) also made clear that there was a deep connection between the topology of S, the positivity of the Arnowitt-Deser-Misner (ADM) mass associated with the given data on S, and the development of black-holes. During the years, Brill-Wheeler's analysis has been improved, proving a number of heuristic models, some of which very refined [5], to the effect of describing the above mechanism in more general settings. None of these models, however, is able to deal with the interplay between topology and matter in the full theory, neither it allows us to get much information on this interplay when S does not support asymptotically Euclidean data (e.g. when S is closed).

The purpose of this article is a tentative to fill this gap by providing an analysis of the interaction between the topology of physical space and its energetic content in the exact theory. This analysis relies on the conformal approach to the initial value problem in general relativity, and on some recent results on minimal surface theory for Riemannian manifolds. In this way we obtain a consistent framework for describing the above interaction without making use of any heuristic model, or without referring to particular classes of initial data.

The paper is organized as follows. In §2 we review the (3 + 1)-(space + time)-description of general relativity and the conformal approach to the initial value problem. In §3 we discuss the interaction between topology and the energetic content of S when S is a complete maximal hypersurface supporting data the Cauchy development of which is an asymptotically flat spacetime. We examine what happens to such data as the «energetic content» of S grows. As is known, in such a case, the discussion can be reduced without loss in generality to an examination of the solvability of the vacuum time-symmetric initial value problem. By a rewriting of a theorem of Cantor and Brill [6], it is possible to discuss such solvability as an eigenvalue problem. The eigenvalues in question are associated to any bounded domain $B \subset S$, and their vanishing provides an obstruction to the solvability of the constraints on S. If originally S were topologically trivial (i.e. $S \simeq \mathbb{R}^3$), then it is shown that this vanishing implies that physical space has undergone a topology change: it is no longer modelled on $S \simeq \mathbb{R}^3$. Physical space is now represented by two disjoint families of embeddings: one modelled on the three-sphere $\mathbb{S}^3$ and the other on $\mathbb{S}^2 \times \mathbb{R}^1$ (other topological configurations are also possible). As expected on the ground of the remarks above, the geometries of such embedding are in general singular. Thus we are naturally led to discuss if in a physical space on the verge of such topological bifurcation (or soonafter than that has occurred), there can develop apparent horizons. For, modulo the validity of the cosmic censorship conjecture, the existence of apparent horizons implies the existence of an event horizon in the Cauchy development of the given data [7]. And is such a case the singularities consequence of the topological bifurcation will do no harm being hidden from observation. Since the (outermost) apparent horizon for time-symmetric data is represented by a minimal two--surface in the physical space, we apply known theorems on the properties of these surfaces to discuss the development of such horizons. It is shown that, generically, apparent horizons do form, but in general not simultaneously to the development of singularities. Notice that such results extend to the full theory the conclusions of a recent elegant heuristic model of Cantor and Piran [5].

In §4 we discuss the interaction between the topology of physical space and its energetic content when S is closed (i.e. compact and without boundary). In this case there are no particular obstructions to the solvability of the constraints susceptible of topological interpretation as before. Here it is the sign of the scalar curvature (actually the sign of the averaged scalar curvature) of the physical space that plays a dominant role. Through the constraints this sign is related to the characteristics of the sources present. In this way, on using again the conformal approach, eigenvalues techniques, and minimal surface theory (through the results of Schoen and Yau [8]), it is possible to show that if a sufficient amount of energy is present (matter plus gravitational radiation) then topology of physical space is that of a three-sphere $\mathbb{S}^3$ (possibly quotiented by a finite group of isometries, G, acting without fixed points) or that of a closed space admitting a countable number of $(S^2 \times S^1)$-wormholes. It also comes out that other topologies such as $S \simeq T^3$, the three-torus topology, restrict a priori the otherwise allowable sets of initial data that can be supported by S. Some of the results of this paragraph have already appeared in [9] (where, however, the approach is more cumbersome owing to a technical error in deriving formula (22) of that paper). In §5 we conclude with some final remarks on the possibility of a dynamically induced topology change in S. As already remarked these changes are accompanied either by a causality violation (a typical example of this behaviour is the Taub-NUT solution [10]) or by the development of a singularity of some sort. This fact has strong implications on any attempt to introduce in the theory the possibility of topology fluctuations at a quantum level and leads to the issue of the validity of Wheeler's spacetime foam picture as recently emphasized by B. DeWitt [1].

Some remarks about notation. If not otherwise specified, all tensor fields are considered in their completely covariant representation. Furthermore for any smooth symmetric bilinear form $\underline{A}$ on a three-dimensional Riemannian manifold $(S, \underline{g})$, we set $\operatorname{tr} \underline{A} \equiv A^i_i$, $\underline{A} \circ \underline{A} \equiv A^{ik} A_{ik}$, $(\nabla \circ \underline{A})_k \equiv \nabla^i A_{ik}$, where ∇ denotes the Riemannian connection associated with g and where $i, k, \ldots, = 1, 2, 3$.

Whenever we need to consider spacetime quantities we use a superscript (4), e.g. ${}^{(4)}\underline{g}$ for a Lorentzian metric, and Greek indices $\alpha, \beta, \ldots, = 1, 2, 3, 4$. Riemann tensor sign conventions are fixed by $(\nabla_i \nabla_k - \nabla_k \nabla_i) v^j = R_{ikl}{}^j v^l$ with $R_{ik} \equiv R^l{}_{ilk}$; physical units are taken so that $G = c = 1$. We will also need to consider a few basic facts about spaces of functions on a given manifold $(S, \underline{g})$. Thus, let $C_0^\infty(S)$ be the space of smooth functions on $(S, \underline{g})$ with compact support and let $E_p(S)$ be the vector space of smooth functions on S that along with their gradients are L_p $(p \geqslant 1)$ summable. For any $f \in E_p(S)$ define the usual H_1^p Sobolev norm as

$$\|f\|_{H_1^p} \equiv \left(\int_S f^p \mathrm{d}v_{\underline{g}}\right)^{1/p} + \left(\int_S (\nabla f \cdot \nabla f)^p \mathrm{d}v_{\underline{g}})^{1/p}\right).$$

Completion of $E_p(S)$ with respect to this norm yields the Sobolev space $H_1^p(S)$. We define similarly the Sobolev spaces $H_s^p(S)$ for $s > 1$. As is known such definitions naturally extend (e.g. via local trivializations) to spaces of mappings between manifolds, $f: S \to V$, giving rise to the Sobolev spaces $H_s^p(S, V)$. For details see for instance [11], [12].

2. THE CONFORMAL APPROACH TO THE INITIAL VALUE PROBLEM: A CURSORY LOOK

Let $(V, {}^{(4)}\underline{g})$ be a spacetime manifold solution of the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, describing a given gravitating system. According to the $(3 + 1)$ dynamical formulation of general relativity, we regard $(V \overset{\varphi}{\simeq} S \times I, {}^{(4)}\underline{g})$ as the Cauchy development of some regular initial data set $(S, \underline{g}, \underline{K})$, where S is a three-manifold carrier of the initial data, φ is a diffeomorphism mapping V onto the product $S \times I$ (I being a suitable subset of $\mathbb{R}$), and $\underline{g}$, $\underline{K}$ are tensor fields on S respectively representing the first and the second fundamental form associated with the embedding $i: S \to V$ of S in the final spacetime. To implement this picture, let $\mathrm{Emb}(S, V, {}^{(4)}g)$ be the set of the emebddings $i: S \to V$ such that $i(S) \subset V$ is everywhere spacelike for ${}^{(4)}g$, and i, as a mapping, belongs to a given Banach space B. If B is suitably chosen (e.g. $H_q^p(S, V)$ for S closed), then $\mathrm{Emb}(S, V, {}^{(4)}g)$ is a smooth (∞-dimensional) Banach manifold the tangent space of which, at the generic embedding i, is the set of vector fields in V covering i. Let $i_t : [-\epsilon, \epsilon] \to \mathrm{Emb}(S, V, {}^{(4)}\underline{g})$ be a curve of embeddings. Such curve defines either a slicing S_t of $(V, {}^{(4)}\underline{g})$ providing a time function in V, or the non-spacelike congruence Γ along which we evolve the data from a slice S_t to the nearby slice $S_{t+\delta t}$. Corresponding to a choice of the curve i_t, we denote by $\underline{n}$ the unit timelike future pointing normal vector field associated with the slicing $\{S_t\}$, by N_t (which is the lapse function) the proper-time normal separation between two nearby slices of $\{S_t\}$, and by t the non-spacelike future pointing vector field (tangent to the t-parametrized lines defining the congruence Γ) covering i_t. As is known [13], $\underline{n}$, N_t, and $\underline{t}$ are related to each other via

$$\underline{t}(x) = (T_x i_t) \circ \underline{\beta}(x) + N_t(x)\underline{n}(i_t(x)), \tag{2.1}$$

where $\underline{\beta} : S_t \to TS_t$ is the shift vector field, and Ti_t is the tangent mapping associated with i_t (henceforth, with a slight abuse of notation we shall write (2.1) as $\underline{t} = \underline{\beta} + N_t \underline{n}$). Notice that as long as N_t is strictly positive, $\underline{t}$ is nowhere tangent to S_t, and the diffeomorphism φ mapping $S \times I$ onto a neighbourhood of $i_0(S) \subset V$ is explicitly provided by $\varphi : [-\epsilon, \epsilon] \times S \to V, (t, x) \mapsto i_t(x)$.

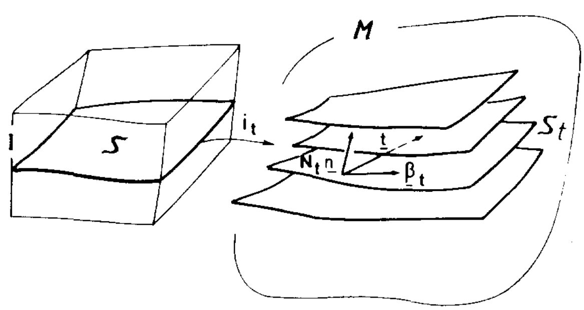

Figure 1. The lapse and shift decomposition associated with the curve of embeddings $i_t : I \to \to \mathrm{Emb}(S, M, {}^{(4)}\underline{g})$.

Let us respectively denote by g_t and $\underline{K}_t$ the riemannian three-metric and the second fundamental form induced on S by the embedding i_t. $\underline{\tilde{K}}_t$ will be the trace-free part of $\underline{K}_t$ (i.e., the shear tensor), and $k_t \equiv \mathrm{tr}\,(\underline{K}_t)$ the mean extrinsic curvature of $i_t(S)$ in V. In terms of the fields $N_t, \underline{n}, \underline{t}, \underline{\beta}$ introduced above, and in the local spacetime coordinates induced on V by φ, we can write

$$^{(4)}\underline{g} = (\underline{g}_t)_{ik}\,\mathrm{d}x^i \mathrm{d}x^k + 2(\underline{\beta}_t)_i\,\mathrm{d}x^i \mathrm{d}t - (N_t^2 - |\underline{\beta}|^2)\,\mathrm{d}t^2, \tag{2.2}$$

$$\underline{K}_t = -\frac{1}{2}\,\pounds_{\underline{n}}(\underline{g}_t)_{ik}\,\mathrm{d}x^i \mathrm{d}x^k = -\frac{1}{2}\,N_t^{-1}(\pounds_{\underline{t}}(\underline{g}_t)_{ik} - \pounds_{\underline{\beta}}(\underline{g}_t)_{ik})\,\mathrm{d}x^i \mathrm{d}x^k, \tag{2.3}$$

$$k_t = N_t^{-1}(\nabla \cdot \beta - \pounds_{\underline{t}}(\log(\det g_t)^{1/2}). \tag{2.4}$$

where £ denotes Lie differentiation along the vector field indicated.

As is well known [13], [14], the initial data $\underline{g}$, $\underline{K}$, together with the data phenomenologically characterizing the given sources: relative mass density μ and momentum density $\underline{J}$ (relative to a set of ∞^3 observers istantaneously at rest on $i_0(S)$), are not freely specifiable, but must satisfy four constraints equations, the Hamiltonian and the divergence constraints, respectively:

$$R(\underline{g}) + k^2 - \underline{K} \cdot \underline{K} = 16\pi\mu, \tag{2.5}$$

$$\nabla \cdot (\underline{K} - k\underline{g}) = 8\pi \underline{J}, \tag{2.6}$$

where $R(\underline{g})$ is the scalar curvature associated with $\underline{g}$, and where the physical admissibility of the sources considered is ensured by requiring that $\mu \geqslant |\underline{J}|$ (the dominant energy condition).

The effect of the above constraints on the possible initial data sets $(\underline{g}, \underline{K})$ is best understood if we recall some basic facts about the geometrical meaning of the scalar curvature. To this end let $\exp_x : T_x S \to S$ denote the exponential mapping associated with $(S, \underline{g})$ at the generic point x, and let us suppose that the injectivity radius ρ of $(S, \underline{g})$ is different from zero (ρ is the largest $\rho > 0$ such that $\exp_x$ is, for every $x \in S$, a diffeomorphism from the open euclidean ball of radius ρ in $T_x S$ onto its image). Let $\bar{B}(x, r)$ denote the ball of radius r ($r < \rho$) in $T_x S$, centered at x, and let $B(x, r)$ be its image on S as obtained via the action of $\exp_x$. If $\Theta(x, r)$ denotes the ratio between the standard euclidean measure of $\bar{B}(x, r)$ and the actual riemannian measure in $(S, \underline{g})$ of $B(x, r)$, i.e. $\Theta(x, r) \equiv \equiv \mathrm{Vol}_{\underline{g}}(B(x, r))/\mathrm{Vol}_{\underline{e}}(\bar{B}(x, r))$ (with $\underline{e}$ the euclidean metric on $\mathbb{R}^3$), then [15]

$$R(\underline{g})(x) = \lim_{r \to 0} (1 - \Theta(x, r))/r^2. \tag{2.7}$$

In other words the scalar curvature locally measures the deviation of the riemannian volume of small geodesic balls from the corresponding euclidean volume. In this sense, the Hamiltonian constraint, relate the giving of the physical riemannian measure on S to the instantaneous distribution of matter and gravitational radiation. This observation motivates the use of conformal techniques in discussing the rationale underlying the action of the constraints (2.5), (2.6) on the fields $(\underline{g}, \underline{K}, \mu, \underline{J})$. Within this context we consider the conformal geometry of the physical space, at the given initial instant, as a given datum. For, that geometrical structure is a convenient representative of the two dynamical degrees of freedom, per space point, of the gravitational field. On the other hand, the actual scale geometry of the physical space (i.e. the giving of a smooth volume form on S), at that initial instant, being constrained by (2.5), cannot be arbitrarily provided.

Thus a compromise is reached by providing S with a riemannian structure $(S, \hat{\underline{g}})$ the underlying conformal geometry of which is the given one, but the associated scale geometry of which is rather arbitrary, chose at our convenience (e.g. by imposing that $R(\hat{\underline{g}})$ is some given function. Notice, however, that not every smooth function on S can be realized as the scalar curvature of some metric the conformal structure of which is preassigned [16]. Later on we shall return on this point). The philosophy is to recover the actual scale geometry of the physical space by means of a conformal rescaling $\underline{g} = \Psi^4 \hat{\underline{g}}$ (locally: $(\underline{g})_{ik} = \Psi^4 (\hat{\underline{g}})_{ik}$), where the conformal factor Ψ is determined, as a consequence of the constraints (2.5) and (2.6), by the Lichnerowicz-York (or scale) equation (see e.g. [13])

$$(2.8) \qquad 8\tilde{\Delta}\Psi + (\hat{\underline{A}} \cdot \hat{\underline{A}})\Psi^{-7} + 16\pi\mu\Psi^{-3} - \frac{2}{3}(k^2)\Psi^5 = 0, \qquad \Psi > 0,$$

where $\tilde{\Delta} = \hat{\Delta} - \frac{1}{8} R(\hat{\underline{g}})$ is the conformally covariant Laplacian associated with $\hat{\underline{g}}$ ($\hat{\Delta}$ being the standard Laplace-Beltrami operator corresponding to $\hat{\underline{g}}$). $\hat{\underline{A}}$ is a symmetric bilinear form on S such that $\mathrm{tr}_{\hat{\underline{g}}}(\hat{\underline{A}}) = 0$, $\hat{\nabla} \cdot \hat{\underline{A}} = 8\pi\hat{\underline{J}} + \frac{2}{3}\Psi^6\hat{\nabla}k$, and $\hat{\mu}$, $\hat{\underline{J}}$ are «trial» densities of mass and momentum of the external sources on S, respectively. The physical meaning of the fields $\hat{\underline{A}}$, $\hat{\mu}$, $\hat{\underline{J}}$ so introduced follows by observing that, in describing the external sources and possible gravitational excitations, we cannot directly use μ, $\underline{J}$, and $\underline{K}$. For, intensive parameters, such as densities, rely for their definition on the scale geometry of their ambient space, and we do not know this latter in advance. Furthermore, the physical fields realizing the external sources may have a non-trivial behaviour under conformal rescaling (e.g. an electromagnetic field or a neutrino flied [17]). These difficulties are overcome by describing the sources and the gravitational excitations on S by means of the «trial» density of mass $\hat{\mu}$, trial density of momentum $\hat{\underline{J}}$, and trial shear $\hat{\underline{A}}$, introduced above. Notice that $\hat{\mu}$ and $\hat{\underline{J}}$ are free data, as well as is a free datum the initial rate of volume expansion k (it provides the initial rate of variation of the scale geometry of $(S, \hat{\underline{g}})$ and it is essentially a kinematical datum, corresponding to a choice of how $(i_0(S), \hat{\underline{g}})$ is embedded in the final spacetime [13]). Notice also that $\hat{\underline{A}}$ is partly constrained by

$$(2.9) \qquad \hat{\nabla} \cdot \hat{\underline{A}} = 8\pi\hat{\underline{J}} + \frac{2}{3}\Psi^6\hat{\nabla}k,$$

a rewriting of the momentum constraint (2.6). It is rather its $\hat{\underline{g}}$-transverse part $\hat{\underline{A}}_\perp$ ($\hat{\nabla} \cdot \hat{\underline{A}}_\perp = \underline{0}$) that is a free datum, describing the initial rate of variation of the conformal geometry of $(i_0(S), \hat{\underline{g}})$.

$(\hat{\underline{g}}, \hat{\underline{A}}_\perp, \hat{\mu}, \hat{\underline{J}}, k)$ are the freely specifiable parts of the initial data set $(\underline{g}, \underline{K}, \mu, \underline{J})$

(the York data associated with $(\underline{g}, \underline{K}, \mu, \underline{J})$ [13], [18]). Such York data are connected to the physical data $(\underline{g}, \underline{K}, \mu, \underline{J})$ by assuming

$$\underline{g} = \Psi^4 \hat{\underline{g}}, \underline{K} = \Psi^{-2} \hat{\underline{A}} + \frac{1}{3} \Psi^4 k \hat{\underline{g}}, \mu = \Psi^{-8} \mu, \underline{J} = \Psi^{-6} \hat{\underline{J}}, \tag{2.10}$$

where the conformal factor Ψ and the bilinear form $\hat{\underline{A}}$ are the solution of the coupled equations (2.8) and (2.9) associated with the given York data set. It is very important to realize that, since the fields $\hat{\underline{A}}$, $\hat{\mu}$, $\hat{\underline{J}}$ are referred to the fictious scale geometry associated with $(S, \hat{\underline{g}})$, York data are defined only in conformal equivalence classes. Namely $\tilde{\underline{g}} = \alpha^4 \hat{\underline{g}}$, $\tilde{\underline{A}} = \alpha^{-2} \hat{\underline{A}}$, $\tilde{\underline{J}} = \alpha^{-6} \hat{\underline{J}}$, $\tilde{\mu} = \alpha^{-8} \hat{\mu}$ (with α a smooth positive function on S) corresponds to the same physical data $(\underline{g}, \underline{K}, \mu, \underline{J})$ as $(\hat{\underline{g}}, \hat{\underline{A}}_{\perp}, \hat{\mu}, \hat{\underline{J}}, k)$ do. In this sense, all the properties of the initial data set $(\underline{g}, \underline{K}, \mu$ $\underline{J})$ and of its Cauchy development only depend on the conformal class the initial data belong to and on the behaviour of the solution of the corresponding Lichnerowicz-York equation (2.8).

3. ASYMPTOTICALLY EUCLIDEAN DATA: TOPOLOGY CHANGES AND THE DEVELOPMENT OF APPARENT HORIZONS

As is well known, the solvability conditions for (2.8) are quite different according to whether S is an open or a closed (i.e. compact and without boundary) manifold. Correspondingly, there are different kind of topological information that we can get from the constraints in these two cases. Let us consider first the case of asymptotically euclidean data.

Let us assumed that $S \simeq \mathbb{R}^3$, and suppose that the fields $(\underline{g}, \underline{K}, \mu, \underline{J})$ satisfy the usual fall-off conditions at spatial infinity associated with the boundedness of the total four-momentum of the gravitational configuration under study. Namely, denoting by r the euclidean distance in the region external to some compact set B, we require that there exist in $S \setminus B$ asymptotic cartesian coordinate charts in which

$$\begin{aligned} &D^a[(\underline{g})_{ik} - e_{ik}] = 0(r^{-1-|a|}), \quad D^a[(\underline{K})_{ij}] = 0(r^{-2-|a|}), \\ &D^a \mu = 0(r^{-4-|a|}), \quad D^a[(\underline{J})_i] = 0(r^{-4-|a|}), \end{aligned} \tag{3.1}$$

where a is a multi-index (i.e. a triple of non-negative integers (a_1, a_2, a_3), with $|a| \equiv (a_1 + a_2 + a_3)$, and $D^a = \partial_1^{a_1} \partial_2^{a_2} \partial_3^{a_3}$). [Usually in going through the proofs of existence and uniqueness for solutions of (2.8), (2.9), it is more convenient to rewrite the above asymptotic conditions in terms of the weighted Sobolev spaces of Niremberg-Walker-Cantor [6], [11], [14]. Since a detailed examination of such proofs will not be strictly necessary in the following, and since the resulting formalism would somewhat obscure the discussion, we spare, the reader that transcription]. We further assume that $(S, \underline{g})$ can be harmonically embedded in the final spacetime (i.e. $k = 0$, $i_0(S)$ in maximal), so that equations (2.8) and (2.9) decouple and investigating the existence of asymptotically euclidean initial data reduces to a discussion of the solvability of the Lichnerowicz-York equation (2.8) (as is known, the solvability of equation (2.9), determining the longitudianl part of $\hat{\underline{A}}$, causes no problems. Notice also the R. Bartnik has recently proven that asymptotically euclidean maximal slices in asymptotically flat spacetimes exist under very general conditions [19], [20]).

Under the above hypotheses, M. Cantor [18] and indipendently Chaljub-Simon and Y. Choquet-Bruhat [21] have provided the existence and uniqueness theorem for problem (2.8). In particular, Cantor has shown that (2.8) is solvable (and thus there exist York data for the given gravitational configuration) if and only if the chosen base metric $\hat{\underline{g}}$ can be conformally deformed, within the same asymptotic class, to a three-metric with non-negative scalar curvature. The solution of (2.8) is the unique and depends smoothly on the given York data. The deformability condition appearing in Cantor's theorem is equivalent to requiring that $\hat{\underline{g}}$ must be conformally deformable to another asymptotically euclidean metric with zero scalar curvature. As is known [13], [18], the existence of such a metric, $\underline{g}^+$, is equivalent to finding a solution to the second order linear elliptic problem

$$(3.2)\qquad \begin{cases} \hat{\Delta}\varphi - \dfrac{1}{8} R(\hat{\underline{g}})\varphi = 0, \\ \varphi > 0 \\ D^a(\varphi - 1) = 0(r^{-1-|a|}), \end{cases}$$

where φ is the conformal factor defining the deformation ($\underline{g}^+ = \varphi^4 \hat{\underline{g}}$). In this connection, Cantor and Brill [6] have shown that problem (3.2) is solvable if and only if, for all functions $f \neq 0$ with compact support, we have

$$(3.3)\qquad -\int_S R(\hat{\underline{g}}) f^2 \, dv_{\hat{\underline{g}}} < 8 \int_S |\nabla f|^2 \, dv_{\hat{\underline{g}}}.$$

This condition ensures that the solvability of the Lichnerowicz-York equation is possible either if $R(\hat{\underline{g}}) \geqslant 0$, or if $R(\hat{\underline{g}})$ is not too negative in a suitable averaged sense, namely if

$$(3.4)\qquad \left(\int_{R(\hat{\underline{g}})<0} |R(\hat{\underline{g}})|^{3/2} \, dv_{\hat{\underline{g}}} \right)^{2/3} \leqslant 8/C,$$

C being a large constant. The obstruction to the solvability of (2.8) (hence to the existence of asymptotically flat data satisfying (2.5), (2.6)) represented by the curvature restriction (3.3) of (3.4) has a suggestive physical interpretation. Roughly spaking, it corresponds to the fact that there cannot be regular, horizon-free, asymptotically euclidean data on $S \simeq \mathbb{R}^3$ supporting too much negative gravitational binding energy. This circumstance has been well-known to workers in relativity since Brill's pioneering work on the time-symmetric, axially-symmetric initial value problem [3]. Following him and Wheeler's remarks [4], we can illustrate heuristically the meaning of (3.4) by considering the case of vacuum time--symmetric initial data sets (i.e., we assume $\mu = 0$, $\underline{K} = \underline{0}$ on S; such data correspond to a momentarily static gravitational configuration). Fix the attention on a smooth sequence of (base) three-metrics $\underline{\hat{g}}$ such that (3.4) is eventually not satisfied. Correspondingly, the conformal factor solution of problem (3.2) (to which the Lichnerowicz-York equation reduces for vacuum time-symmetric data) tends to vanish on some (topologically) spherical surface enclosing the region of increasing gravitational energy. Eventually, this behaviour of the conformal factor leads to the development of a singularity in the physical geometry $(S, \underline{g})$ (a «bag of gold» singularity) which may or may not be hidden, from an observer at infinity, by an apparent horizon. The correctness of such picture has been confirmed also by various numerical calculations (e.g., see Eppley [22]), and by a number of heuristic examples which try to model (following the original remarks by Brill and Wheeler) the mechanism above by examining the solution of the elliptic problem $(\Delta_{\underline{e}} - H) f = 0, f > 0$, in euclidean space (with $f \rightarrow 1$ asymptotically or in the origin, and where H is a radial step function), the most refined model in this direction being the description proposed by Cantor and Piran [5]. *It is worth emphasizing that although such models capture the essential features underlying the curvature obstruction (3.4), they do not directly refer to the problem (3.2) with its rich underlying geometrical meaning*. Thus, in this connection, it is interesting that on applying some recent results by R. Schoen and D. Fischer-Colbrie [23] (results referring to the classification of stable minima surfaces in complete three-dimensional manifolds), we are able to describe the above topology-change mechanism in the full theory without recourse to any heuristic model.

The first eigenvalue of the conformal laplacian

Again, we refer to vacuum time-symmetric data, for, as a consequence of Cantor's theorem, the solvability of the Lichnerowicz-York equation in the generic case (on a maximal slice) is assured if and only if the associated vacuum time-symmetric initial value problem is solvable. Thus we are led to discuss problem (3.2) trying to interpret the solvability condition (3.3). To this end, let $B \subset S$ by any given bounded domain, and let $\lambda_1(B) < \lambda_2(B) < \lambda_3(B) \ldots$ be the sequence of eigenvalues of the elliptic operator $\left(\hat{\Delta} - \frac{1}{8} R(\underline{\hat{g}})\right)$, acting on smooth functions vanishing on ∂B. That is, the λ's such that there exist $f \not\equiv 0$, with $\operatorname{supp} f \subset B$ and

$$\left[-\Delta + \frac{1}{8} R(\underline{\hat{g}})\right] f = \lambda f. \tag{3.5}$$

We will be particularly interested to the first eigenvalue $\lambda_1(B)$ of (3.5), the variational characterization of which is

$$\lambda_1(B) = \operatorname{Inf} \left\{ \int_B \left(|\hat{\nabla} f|^2 + \frac{1}{8} R(\underline{\hat{g}}) f^2 \right) \mathrm{d}v_{\underline{\hat{g}}}; \right. \tag{3.6}$$
$$\left. \operatorname{supp} f \subset B, \int_B f^2 \, \mathrm{d}v_{\underline{\hat{g}}} = 1 \right\}.$$

As is well known, $\lambda_1(B)$ has multiplicity one, and the corresponding eigenfunction f_1 does not change sign, so that we can assume $f_1 > 0$ on B. $\lambda_1(B)$ depends continuously on B, and if B, B' are connected domains in S with $B \subsetneqq B'$ then $\lambda_1(B) > \lambda_1(B')$. Our interest in $\lambda_1(B)$ rests on the observation (essentially due to J.Kazdan and F. Warner [24]) that the sign of $\lambda_1(B)$ is a conformal invariant. That is, it only depends on the conformal structure associated with the riemannian manifold $(S, \underline{\hat{g}})$. In particular, if $\underline{\hat{g}}$ is the base metric associated with a given York data set, and $\underline{g}$ is the physical metric obtained from $\underline{\hat{g}}$ by solving the Lichnerowicz-York equation ($\underline{g} = \Psi^4 \underline{\hat{g}}$, with Ψ solution of (2.8)), then we must necessarily have

$$\operatorname{sign} \lambda_1(B; \underline{g}) = \operatorname{sign} \lambda_1(B; \underline{\hat{g}}), \tag{3.7}$$

for each connected domain $B \subset S$ (henceforth, we write $\lambda_1(B; \underline{g})$ etc. whenever we need to emphasize which riemannian structure has been used in evaluating sign $\lambda_1(B)$). Relation (3.7) is quite relevant to our analysis, so that is appropriate to spend few lines in deriving it. First, it is convenient to rewrite (3.6) as a Raleigh-Ritz quotient, namely as

$$\lambda_1(B; \underline{\hat{g}}) = \operatorname{Inf} J(B; \underline{\hat{g}}, f),$$

where

$$J(B;\underline{\hat{g}},f) \equiv \left(\int_B f^2 \mathrm{d}v_{\underline{\hat{g}}}\right)^{-1}\left[\int_B \left(|\hat{\nabla} f|^2 + \frac{1}{8} R(\underline{\hat{g}}) f^2\right) \mathrm{d}v_{\underline{\hat{g}}}\right].$$

Now let $\underline{g} = \psi^4 \underline{\hat{g}}$, and use Lichnerowicz's formula

$$R(\underline{g})\Psi^5 = -8\hat{\Delta}\Psi + R(\underline{\hat{g}})\Psi, \tag{3.8}$$

to get, after some integrations by parts,

$$J(B;\underline{g},f) = \left(\int_B f^2\Psi^6 \mathrm{d}v_{\underline{\hat{g}}}\right)\left(\int_B f^2\Psi^2 \mathrm{d}v_{\underline{\hat{g}}}\right) J(B;\underline{\hat{g}},(\Psi f)\ ,$$

which yields (3.7).

The connection between the properties of $\lambda_1(B)$ and the solvability of problem (3.2) is contained in a theorem proved by D. Fischer-Colbrie and R. Schoen [23] stating that (3.2) *admits a solution if and only if*

$$\lambda_1(B;\underline{\hat{g}}) > 0 \tag{3.9}$$

for every bounded domain $B \subset S$. Clearly this result is, in the present context, simply a restatement of Cantor's theorem, since $\lambda_1(B;\underline{\hat{g}}) > 0$ if and only if condition (3.3) (or, equivalently (3.4)) holds true. The advantage of this formulation lies in the fact that now the curvature obstruction (3.3) can be simply interpretated in terms of the conformal invariance of sign $\lambda_1(B)$. For, if the hamiltonian constraint (2.5) is to be solvable on a maximal slice S, then necessarily $R(\underline{g}) \geqslant 0$ for the physical metric $\underline{g}$. This in turn implies that sign $\lambda_1(B;\underline{g}) > 0$ for every $B \subset S$. Hence, owing to the conformal invariance of sign $\lambda_1(B;\underline{g})$, we get sign $\lambda_1(B;\underline{\hat{g}}) > 0$, $\forall B \subset S$, as a necessary condition for the solvability of (2.9) for a given base metric $\underline{\hat{g}}$ (for the sufficiency so such condition the proof proceeds as in Cantor [18]).

Topological bifurcations

Let us see what happens to the geometry of the physical space $(S, \underline{g} = \varphi^4 \underline{\hat{g}})$ when, by deforming the base metric $\underline{\hat{g}}$, we get $\lambda_1(B;\underline{\hat{g}}) \to 0$ for some domain $B \subset S$. To this end let $\underline{\hat{g}}(\beta)$, $\beta \in (0, 1)$, denote a family of metrics depending analytically on a parameter β, and let $\underline{\hat{g}}(0) = \underline{\hat{g}}$ the given base metric for which problem (3.2) is assumed to have a solution. Let us chose a flow $\beta \to \underline{\hat{g}}(\beta)$ and a domain $B \subset S$ for which $R(\underline{\hat{g}}(\beta))$ becomes more and more negative as β increases. For β sufficiently small we can assume that $\lambda_1(B;\underline{\hat{g}}(\beta)) > 0$ for every $B' \subset S$, so that the problem

$$
(3.10) \qquad \begin{cases} \hat{\Delta}_\beta \varphi_\beta - \dfrac{1}{8} R(\hat{\underline{g}}(\beta))\varphi_\beta = 0 \\ \varphi_\beta > 0, \\ D^a(\varphi_\beta - 1) = 0(r^{-1-|a|}) \end{cases}
$$

admits a smooth positive solution φ_β. With these premises let $f_1(\beta)$ be the first eigenfunction (see (3.5)) associated with $(B;\hat{\underline{g}}(\beta))$. Namely the $f_1(\beta)$ such that

$$
(3.11) \qquad \begin{cases} \left[-\hat{\Delta}_\beta + \dfrac{1}{8} R(\hat{\underline{g}}(\beta))\right] f_1(\beta) = \lambda_1(B, \hat{\underline{g}}(\beta)) f_1(\beta) \\ f_1(\beta)|_{\partial B} = 0, \quad \displaystyle\int_B f_1^2(\beta) \mathrm{d}v_{\hat{\underline{g}}(\beta)} = 1. \end{cases}
$$

Multiplying equation (3.10) with $f_1(\beta)$' integrating over B the resulting expression, and on applying Green's theorem we get

$$
\int_B \varphi_\beta \hat{\Delta}_\beta f_1(\beta) \mathrm{d}v_{\hat{\underline{g}}(\beta)} - \int_{\partial B} \varphi_\beta (\hat{\nabla} f_1(\beta) \circ \hat{\underline{s}})\, \mathrm{d}\hat{\sigma} -
$$

$$
- \frac{1}{8} \int_B \varphi_B R(\hat{\underline{g}}(\beta)) f_1(\beta)\, \mathrm{d}v_{\hat{\underline{g}}(\beta)} = 0
$$

($\hat{\underline{s}}$ being the outward unit normal to ∂B). That is, on taking into account (3.11)

$$
(3.12) \qquad \lambda_1(B, \hat{\underline{g}}(\beta)) \int_B f_1(\beta) \varphi_\beta \mathrm{d}v_{\hat{\underline{g}}(\beta)} = - \int_{\partial B} \varphi_\beta (\hat{\nabla} f_1(\beta) \circ \hat{\underline{s}}) \mathrm{d}\hat{\sigma}.
$$

The term $(\hat{\nabla} f_1(\beta) \circ \hat{s})$ appearing in the surface integral in (3.12) is nonpositive on ∂B [otherwise from $(\hat{\nabla} f_1(\beta) \circ \hat{\underline{s}})|_{\partial B} > 0$ together with $f_1(\beta)|_{\partial B} = 0$ we would get $f_1(\beta) < 0$ in a neighbourhood, $B_\epsilon \subset B$, of ∂B, contradicting the positivy of the first eigenfunction]. More in particular on applying Gauss's theorem to equation (3.11) over B we get

$$
(3.13) \qquad \int_{\partial B} (\hat{\nabla} f_1(\beta) \cdot \hat{\underline{s}}) \mathrm{d}\hat{\sigma} = \int_B \left[\frac{1}{8} R(\hat{\underline{g}}(\beta) - \lambda_1(B, \hat{\underline{g}}(\beta))\right] f_1(\beta) \mathrm{d}v_{\hat{\underline{g}}(\beta)}
$$

which shows that in our case (since $R(\hat{\underline{g}}(\beta)) < 0$), $(\hat{\nabla} f_1(\beta) \circ \hat{\underline{s}})|_{\partial B}$ can be taken strictly negative. Thus we can rewrite (3.12) as

$$\lambda_1(B, \hat{\underline{g}}(\beta)) \int_B f_1(\beta) \varphi_\beta \, \mathrm{d} v_{\hat{\underline{g}}(\beta)} = \int_{\partial B} \varphi_\beta \, | \hat{\nabla} f_1(\beta) \circ \hat{\underline{s}} | \, \mathrm{d} \hat{\sigma}. \tag{3.14}$$

There are two different regimes in which we can discuss the implications of this relation. In the first case we assume that while deforming the original metric via the flow $\beta \to \hat{\underline{g}}(\beta)$, the volume of the region B, in the physical metric $\underline{g} = \varphi_\beta^4 \hat{\underline{g}}(\beta)$, remains uniformly bounded above:

$$(\mathrm{Vol}\,(B, \underline{g} = \varphi_\beta^4 \hat{\underline{g}})) \leqslant C, \qquad \forall \beta. \tag{3.15}$$

Under this assumption (3.14) implies that if $\lim_{\beta \to 1} \lambda_1(B, \hat{\underline{g}}(\beta)) = 0$, then necessarily $\lim_{\beta \to 1} \varphi_\beta|_{\partial B} = 0$; namely ∂B becomes a nodal surface for φ. Thus in this case we do not have any longer global solvability for problem (3.10) on the whole original model manifold $S \simeq R^3$. Rather we may have solvability of problem (3.10) restricted to the region B and to the region $S \setminus B$. In B it is easily checked that

$$\lim_{\beta \to 1} \varphi_\beta = C' f_1(\beta = 1) \tag{3.16}$$

where the constant C' is fixed by the condition (3.15). While in the external region $S \setminus B$ $\zeta \equiv \lim_{\beta \to 1} \varphi_\beta$ will be well defined if there exists a smooth solution to the problem

$$\begin{cases} \left[-\hat{\Delta}_\beta + \dfrac{1}{8} R(\hat{\underline{g}}(\beta)) \right]_{\beta = 1} \zeta = 0, \qquad \zeta > 0 \quad \text{in } S \setminus B \\ \zeta|_{\partial B} = 0 \\ D^a(\zeta - 1) = 0(r^{-1-|a|}), \end{cases} \tag{3.17}$$

Again, this is solvable if and only if for each domain $B' \subset (S \setminus B)$, $\lambda_1(B', \hat{\underline{g}}(\beta = 1)) > 0$. Assuming that the deformation $\hat{\underline{g}} \to \hat{\underline{g}}(\beta)$ is such that also this last condition holds true, we can easily interpret what happens to the geometry of the physical space when $\lambda_1(B, \hat{\underline{g}}) \to 0$. For as follows from the above results, we can describe the conformal mapping $\hat{\underline{g}}(\beta) \to \underline{g} = \varphi_\beta^4 \hat{\underline{g}}(\beta)$, as $\beta \to 1$, as a compactification of the region B into a closed physical space $(S_C \simeq \mathbf{S}^3, \underline{g})$. Its geometry is provided (up to a constant scale factor, see (3.16)) by $\underline{g} = f_1^4(\beta) \hat{\underline{g}}(\beta)|_{\beta = 1}$, and it is «singular» around the «point» that corresponds to the surface ∂B on which $f_1 = 0$. Similarly, the region $S \setminus B$ is mapped by $\hat{\underline{g}}(\beta) \to \underline{g} = \varphi_\beta^4 \hat{\underline{g}}(\beta)$, as $\beta \to 1$, into an asymptotically euclidean three-manifold $(S_0 \simeq \mathbf{S}^2 \times R^1, \underline{g} = \zeta^4 \hat{\underline{g}}(\beta)|_{\beta = 1})$, with ζ solution of problem (3.17). Also in this $(S_0, \underline{g})$ the surface

∂B is identified with a single «point» around which the geometry associated with $\underline{g}$ is «singular» (the «bag of gold» singularity of Wheeler [4]).

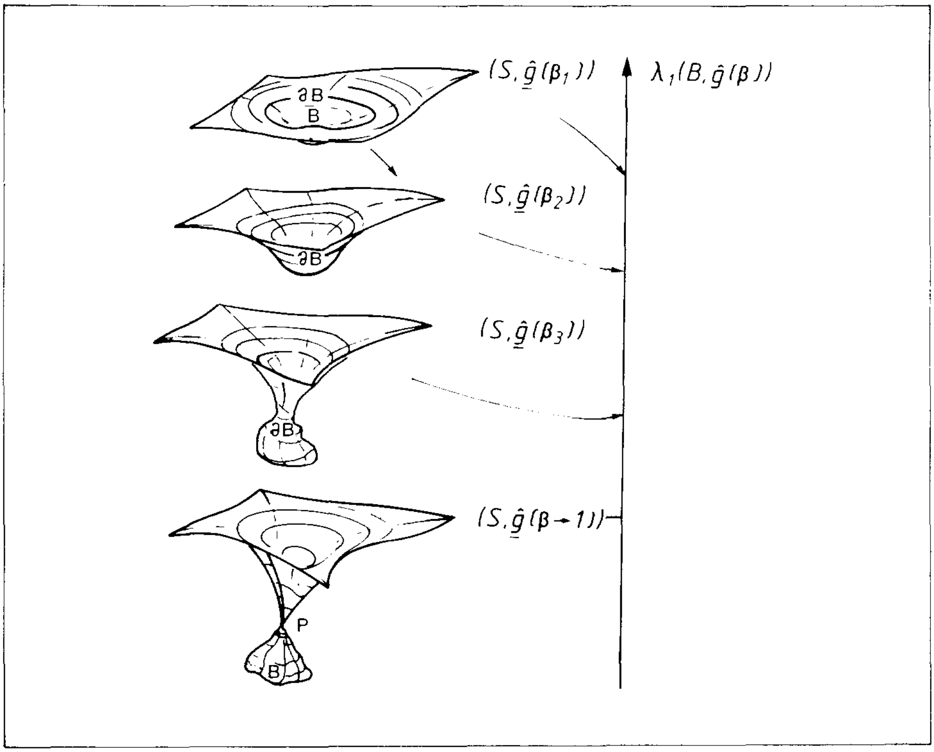

Figure 2. Compactification of the region B into a closed physical space. As $\beta \to 1$, and $\lambda_1(B, g(\beta)) \to 0$, $\partial B \subset (S, g)$ shrinks to zero, and there is no regular physical geometry that can describe a neighborhood of ∂B in $(S, \underline{g})$.

Notice that a different type of topological change occurs if instead of condition (3.15) we assume that, while deforming the original metric $\hat{\underline{g}}$ via the flow $\beta \to \to \hat{\underline{g}}(\beta)$, the value of φ_β at the boundary ∂B remains uniformly bounded away from zero, e.g.

$$\varphi_\beta|_{\partial B} = 1. \tag{3.18}$$

Under this assumption (3.14) implies that

$$\lambda_1(B, \hat{\underline{g}}(\beta)| \left(\int_B f_1^2(\beta) \mathrm{d} v_{\hat{\underline{g}}(\beta)}\right)^{1/2} \left(\int_B \varphi_\beta^2 \mathrm{d} v_{\hat{\underline{g}}(\beta)}\right)^{1/2} \geqslant \int_{\partial B} |\hat{\nabla} f_1(\beta) \circ \underline{\hat{s}}| \,\mathrm{d}\hat{\sigma},$$

where we have used Hölder inequality. Since $\int f_1^2(\beta) \mathrm{d} v_{\hat{\underline{g}}(\beta)} = 1$, and

$(\int \varphi_\beta^2 \, dv_{\hat{\underline{g}}(\beta)})^{1/2} \leqslant |\operatorname{Vol}(B, \underline{g} = \varphi_\beta^4 \hat{\underline{g}}(\beta))|^{1/6} |\operatorname{Vol}(B, \hat{\underline{g}}(\beta)|^{3/4}$ (again Hölder inequality), we get

$$\lambda_1(B, \hat{\underline{g}}(\beta)) | |\operatorname{Vol}(B, \underline{g} = \varphi_\beta^4 \hat{\underline{g}}(\beta))|^{1/6} |\operatorname{Vol}(B, \hat{\underline{g}}(\beta)|^{3/4} \geqslant \int_{\partial B} |\hat{\nabla} f_1(\beta) \circ \hat{\underline{s}}| \, d\hat{\sigma},$$

which implies that if $\lambda_1(B, \hat{\underline{g}}(\beta)) \to 0$, then necessarily the volume of the region B in the physical metric $\underline{g} = \varphi_\beta^4 \hat{\underline{g}}(\beta)$ blows up.

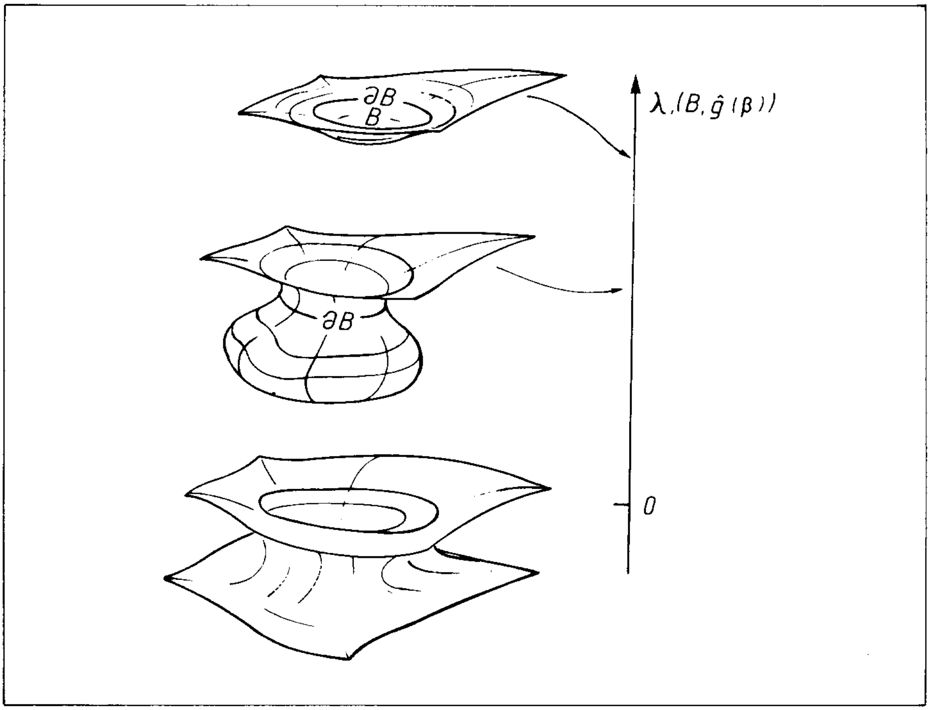

Figure 3. The region B blows up as $\lambda_1(B, g(\beta)) \to 0$ if we assume the boundary condition (3.18).

This last case although interesting is less natural than the situation previosly examined and will be not discussed any further. In effect, condition (3.18) is much more restrictive than (3.15) since it corresponds to considering only deformations of the base metric $\hat{\underline{g}}$ which leave the two-manifold $(\partial B, \hat{\underline{g}})$ fixed. In any case, either if we assume (3.15) or (3.18), as $\lambda_1(B, \hat{\underline{g}}) \to 0$ we have a topological change for the manifold S which is supposed to model the physical space. This fact leads to some interesting conclusions. In particular it suggests that the notion of physical space in general relativity is a little subtler than expected if we adopt the conformal approach for solving the initial value problem. A little reflection may help understanding where the problem is. We start with a given manifold on which it is assumed that the physical space is modelled. However (and here is the point that is generally brushed under the rug), this manifold comes into play, according to the conformal approach, only as the manifold underlying the possible free initial data sets (i.e. the unconstrained part of the initial data) that may possible compete to the gravitational configuration we wish to consider. For instance, in the case discussed above, we were dealing with $\mathbb{R}^3$ endowed with all possible asymptotically euclidean conformal structures it can support. Given any such structure, for instance $(\mathbb{R}^3, \underline{\hat{g}})$, we were only able to say that the corresponding physical space model $(S, \underline{g})$ was a three-manifold S supporting the riemannian structure $\underline{g} = \varphi^4 \underline{\hat{g}}$, with φ solution of problem (3.2). And, as we have seen, under such hypotheses S does not need to be topologically $\mathbb{R}^3$.

What happens can be properly described as the onest of bifurcation phenomena. On one side we have the possible geometrical configurations of the physical space associated to a vacuum, time-symmetric gravitational configuration. On the other side we have as a «parameter space» the set of asymptotically euclidean conformal structures on $\mathbb{R}^3$. This two sets are connected through the Hamiltonian constraint which take the form of the elliptic problem (3.2). Bifurcation occurs as we move in the parameter space so to make $\lambda_1(B, \underline{\hat{g}})$, for some $B \subset \mathbb{R}^3$, pass through zero and become negative. In such case the hamiltonian constraint admits at least two distinct curves of solutions $\underline{\hat{g}}_\lambda \to (S_C, \underline{g} = f_1^4 \underline{\hat{g}}_\lambda)$ and $\underline{\hat{g}}_\lambda \to$ $\to (S_0, \underline{g} = \zeta^4 \underline{\hat{g}}_\lambda)$ which model physical space on the topologically non trivial manifolds $S_C \simeq \mathbb{S}^3$ and $S_0 \simeq \mathbb{S}^2 \times \mathbb{R}^1$.

Arnowitt-Deser-Misner mass and the development of apparent horizons

This nice picture, however, is affected by a serious problem. For, the fact that either $(S_C, \underline{g})$ or $(S_0, \underline{g})$ are singular may suggest that as initial data sets both $(S_C, \underline{g})$ and $(S_0, \underline{g})$ are not physically acceptable, and that the conformal approach may break down in such circumstances. In order to discuss this point, let us fix our attention on the Arnowitt-Deser-Misner mass (see e.g. [25]), $m_{\mathrm{ADM}}(\underline{g})$, associated with the data set $(S, \underline{g})$. In asymptotically cartesian coordinates, the expression for $m_{\mathrm{ADM}}(\underline{g})$ has the familiar form

$$(3.19) \qquad m_{\mathrm{ADM}}(\underline{g}) = \int_{\mathbb{S}_2^\infty} (g_{ij,j} - g_{jj,i}) \, d\sigma_i,$$

where $\mathbb{S}_2^\infty$ is the two-sphere intercepted at spatial infinity by the given slice $(S, \underline{g})$. Similarly, we may formally define $m_{\mathrm{ADM}}(\underline{\hat{g}})$, the mass associated with the given base metric $\underline{\hat{g}}$ [strictly speaking, on using suitable diffeomorphisms and conformal changes in $\hat{\underline{g}}$, we can always choose a base metric $\hat{\underline{g}}$ in such a way as to have $m_{\mathrm{ADM}}(\hat{\underline{g}}) = 0$, see e.g. [13]].

The fact that the metrics $\underline{g}$ and $\hat{\underline{g}}$ are conformally related implies that [13]

(3.20) $$m_{\mathrm{ADM}}(\underline{g}) - m_{\mathrm{ADM}}(\hat{\underline{g}}) = -\frac{1}{2\pi}\int_{S_2^\infty} (\hat{\nabla}^i \varphi)\,\mathrm{d}\sigma_i .$$

To see what happens to this expression when $\varphi \to 0$ somewhere in S, let us introduce the function

(3.21) $$W \equiv \log \varphi,$$

in terms of which equation (3.2) can be rewritten as

(3.22) $$\hat{\Delta} W = \frac{1}{8} R(\hat{\underline{g}}) - |\hat{\nabla} W|^2 .$$

Let us integrate this equation over $(S \setminus Q, \hat{\underline{g}})$, where Q is a compact set in S with smooth boundary ∂Q. On applying Gauss's theorem we get

$$\int_{S_2^\infty} (\hat{\nabla} \log \varphi) \circ \mathrm{d}\hat{\sigma} - \int_{\partial Q} (\hat{\nabla} \log \varphi) \circ \mathrm{d}\hat{\sigma} = \frac{1}{8}\int_{S\setminus Q} (R(\hat{\underline{g}}) - 8\,|\hat{\nabla} W|^2)\,\mathrm{d}v_{\hat{\underline{g}}} .$$

Since $\varphi \to 1$ on S_2^∞, we get

$$-\frac{1}{2\pi}\int_{S_2^\infty} (\hat{\nabla}\varphi) \circ \mathrm{d}\hat{\sigma} = -\frac{1}{2\pi}\int_{\partial Q} (\hat{\nabla} \log \varphi) \circ \mathrm{d}\hat{\sigma} - \frac{1}{16\pi}\int_{S\setminus Q} (R(\hat{\underline{g}}) - 8\,|\hat{\nabla} W|^2)\,\mathrm{d}v_{\hat{\underline{g}}} .$$

That is, on taking into account (3.20),

(3.23) $$m_{\mathrm{ADM}}(\underline{g}) - m_{\mathrm{ADM}}(\hat{\underline{g}}) = -\frac{1}{2\pi}\int_{\partial Q} ((\hat{\nabla} \log \varphi) \circ \underline{\hat{s}})\,\mathrm{d}\hat{\sigma} - \frac{1}{16\pi}\int_{S\setminus Q} (R(\hat{\underline{g}}) - 8\,|\hat{\nabla} W|^2)\,\mathrm{d}v_{\hat{\underline{g}}},$$

where $\underline{\hat{s}}$ is the outward unit normal to ∂Q (thought as an embedded two-surface in $(S, \underline{\hat{g}})$). Let us now deform $\underline{\hat{g}}$ in such a way that somewhere within Q, $R(\underline{\hat{g}})$ gets more and more negative until condition (3.4) breaks down and $\varphi \to 0$ on some nodal two-surface ∂B strictly contained in Q. Notice that $(m_{\mathrm{ADM}}(\underline{g}) - m_{\mathrm{ADM}}(\underline{\hat{g}}))$ keeps track of these changes in Q through the surface integral in (3.23) and through the term $|\hat{\nabla} W|^2$ in the volume integral over $S \setminus Q$. On applying a technique devised by Yau [26] (see also [20]) it can be shown that this latter term is bounded above, and that in particular for any $0 < \beta < 1, p \geqslant 3$, there is an estimate of the form

$$\left[\int_{B((1-\beta)r)} |\hat{\nabla} W|^{2p} \mathrm{d}v_{\underline{\hat{g}}}\right]^{1/p} \leqslant a \left(\int_{B(r)} K^p \mathrm{d}v_{\underline{\hat{g}}}\right)^{1/p} + $$

$$+ \frac{1}{8} a p^{1/2} \left(\int_{B(r)} (R(\underline{\hat{g}}))^p \mathrm{d}v_{\underline{\hat{g}}}\right)^{1/p} + a\beta^{-2} p r^{-2} [\mathrm{Vol}\,(B(r), \underline{\hat{g}})]^{1/p},$$

where a is a numerical constant, K is a non-negative function so that $-K$ is the lower bound of the Ricci curvature of $(S, \underline{\hat{g}})$, and $B(r)$ is the generic geodesic ball of center x_0 and radius r in $(S, \underline{\hat{g}})$. While this estimate can be used to bound the contribution of $|\hat{\nabla} W|^2$ to (3.23) it cannot help in controlling the surface integral in (3.23). For this latter it is more convenient to proceed as follows. Let us consider more closely the integrand appearing in it, namely $(\underline{\hat{s}} \circ \hat{\nabla} \log \varphi)$. If we set

$$s^i \equiv \varphi^{-2} \hat{s}^i,$$

(such $\underline{s}$ is the unit normal, in the physical metric $\underline{g}$, to the surface ∂Q), then a straighforward calculation yields

$$\hat{s}^i \hat{\nabla}_i \log \varphi = \frac{1}{4} \varphi^2 \nabla_i s^i - \frac{1}{4} \hat{\nabla}_i \hat{s}^i. \tag{3.24}$$

Hence (3.23) can be rewritten as

$$\begin{aligned} m_{\mathrm{ADM}}(\underline{g}) - m_{\mathrm{ADM}}(\underline{\hat{g}}) = &- \frac{1}{2\pi} \int_{\partial Q} (\varphi^2 \nabla_i s^i) \mathrm{d}\sigma + \frac{1}{8\pi} \int_Q (\hat{\nabla}_i \hat{s}^i) \mathrm{d}\hat{\sigma} - \\ &- \frac{1}{16\pi} \int_{S \setminus Q} (R(\underline{\hat{g}}) - 8 |\hat{\nabla} W|^2) \mathrm{d}v_{\underline{\hat{g}}}. \end{aligned} \tag{3.25}$$

Notice that the surface integral $\int_{\partial Q} (\hat{\nabla}_i \hat{s}^i) d\hat{\sigma}$ depends only on the choice of the embedding of the surface ∂Q in $(S, \hat{g})$ [for, $(\hat{\nabla}_i \hat{s}^i)$ is simply (minus) the trace of the extrinsic cuvature associated with the embedding of ∂Q in $(S, \hat{g})$]. The troublesome term in (3.25) that still keeps track of the possibly bad behaviour of φ within Q is $\int (\varphi^2 \nabla_i s^i) d\sigma$.

Thus, if we were able to choose a ∂Q for which $\nabla_i s^i|_{\partial Q} = 0$, everything would be fine. Notice that such condition characterizes ∂Q as an extremal two-surface in the physical space $(S, \underline{g})$ (the proof that such ∂Q is actually a stable minimal two-surface is discussed below). Since the data are time-symmetric, that ∂Q would be an example of apparent horizon (an apparent horizon in a general $(S, \underline{g})$ is a closed two-surface with unit normal $\underline{s}$ such that $\nabla_i s^i \pm K_{ij} (g^{ij} - s^i s^j) = 0$, where the plus sign refers to a future apparent horizon, while the minus sign refers to a past apparent horizon). If we assume cosmic censorship, the presence of such ∂Q will indicate the presence, in the Cauchy development of the data $(S, \underline{g})$, of a black hole.

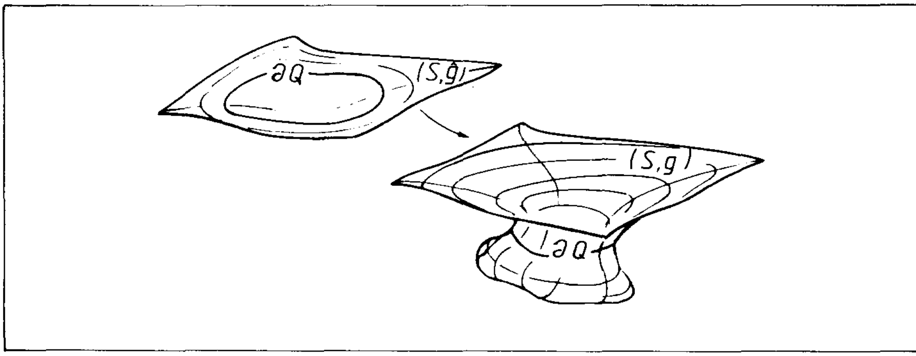

Figure 4. A minimal 2-surface ∂Q in physical space (S, g) image of a closed 2-surface in the base space (S, g).

The topology of apparent horizons

According to the above remarks, we are led to discuss the possibility of choosing ∂Q in such a way as to have $(\nabla_i s^i)|_{\partial Q} = 0$ in the physical space $(S, \underline{g})$. As follows from (3.24), this is the case if the embedding of ∂Q in the given base space $(S, \hat{g})$ can be chosen in such a way as to have

$$\hat{\nabla}_i \hat{s}^i + 4(\hat{s}^i \hat{\nabla}_i \log \varphi) = 0. \tag{3.26}$$

Before discussing the feasibility of this choice, let us first remark that there is a necessary topological requirement that ∂Q must satisfy if, as it will come out, ∂Q is embedded as a stable minimal two-manifold in $(S, \underline{g})$. To this end, and also for future reference, let us first recall the second variation formula for the area functional for hypersurfaces.

Let $\underline{h}$ and $\underline{b}$ respectively denote the first and the second fundamental form associated with the embedding of ∂Q in (S, g). Locally $h_{ij} = g_{ij} - s_i s_j$, $b_{ij} = -h_i^r h_j^p \nabla_r s_p$. If we deform $(\partial Q, \underline{h})$, in $(S, \underline{g})$, along the normal direction $\underline{z} = f\underline{s}$, where f is a smooth function with compact support on ∂Q, then the first variation formula for the area functional provides:

$$\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{Vol}(\partial Q, \underline{h}_z)) = -\int_{\partial Q} \mathrm{tr}(\underline{b}) f \,\mathrm{d}\sigma_z .$$

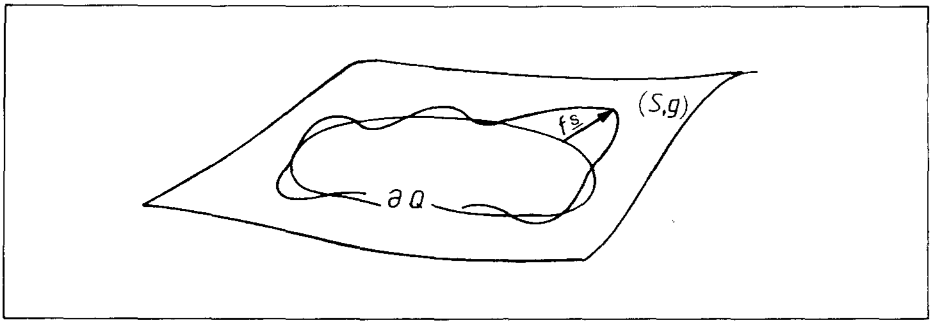

Figure 5. A normal deformation of the embedded 2-surface ∂Q.

from this we get

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\mathrm{Vol}(\partial Q, \underline{h}_z)) = -\int_{\partial Q} f^2[\,|\underline{b}|^2 + \mathrm{Ric}(\underline{s}, \underline{s}) - (\mathrm{tr}(\underline{b}))^2]\,\mathrm{d}\sigma_z - \tag{3.27}$$

$$-\int_{\partial Q} f({}^{(2)}\Delta f)\,\mathrm{d}\sigma_z ,$$

where we have explicitly taken into account the fact that the Lie derivative of $(\mathrm{tr}(\underline{b}))$ along $\underline{z} = f\underline{s}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{tr}(\underline{b})) = [\,|\underline{b}|^2 + \mathrm{Ric}(\underline{s}, \underline{s}) + {}^{(2)}\Delta]\, f, \tag{3.28}$$

with $|\underline{b}|^2 \equiv b_{ij}\, b_{rm} h^{ir} h^{jm}$, and where ${}^{(2)}\Delta$ and $\mathrm{Ric}(\underline{s}, \underline{s})$ respectively are the Laplace-Beltrami operator on ∂Q, associated with $\underline{h}$, and the Ricci curvature, associated with $\underline{g}$, in the direction $\underline{s}$.

Since $f({}^{(2)}\Delta f) = \frac{1}{2}({}^{(2)}\Delta f^2) - |{}^{(2)}\nabla f|^2$, an integration by parts in (3.27) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\mathrm{Vol}(\partial Q, \underline{h}_z)) = \int_{\partial Q} [|{}^{(2)}\nabla f|^2 - (|\underline{b}|^2 + \tag{3.29}$$
$$+ \mathrm{Ric}(\underline{s}, \underline{s}) - (\mathrm{tr}(\underline{b}))^2) f^2] \, \mathrm{d}\sigma_z.$$

From the minimality of $(\partial Q, \underline{h})$ we get $(\mathrm{tr}(\underline{b})) = 0$, so that the stability condition $(\mathrm{d}^2(\mathrm{Vol}(\partial Q, \underline{h}))/\mathrm{d}z^2)_{z=0} \geqslant 0$ yields

$$\int_{\partial Q} [|{}^{(2)}\nabla f|^2 - (\mathrm{Ric}(\underline{s}, \underline{s}) + |\underline{b}|^2) f^2] \, \mathrm{d}\sigma \geqslant 0, \tag{3.30}$$

for any function f with compact support on ∂Q.

Following Schoen and Yau [27], we can rewrite the terms $(\mathrm{Ric}(\underline{s}, \underline{s}) + |\underline{b}|^2)$ appearing in (3.30) in a more useful way. From the contracted Gauss curvature equation we get

$$H = R(g) - 2\,\mathrm{Ric}(\underline{s}, \underline{s}) + (\mathrm{tr}(\underline{b}))^2 - |\underline{b}|^2, \tag{3.31}$$

where H is the scalar curvature of $(\partial Q, \underline{h})$. Since $(\partial Q, \underline{h})$ is minimal, we can rewrite (3.31) as

$$\mathrm{Ric}(\underline{s}, \underline{s}) + |\underline{b}|^2 = \frac{1}{2} R(\underline{g}) - \frac{1}{2} H + \frac{1}{2} |\underline{b}|^2, \tag{3.32}$$

which, when introduced in (3.30), together with $R(\underline{g}) = 0$, yields

$$\int_{\partial Q} [|{}^{(2)}Vf|^2 + \frac{1}{2}(H - |\underline{b}|^2) f^2] \, \mathrm{d}\sigma \geqslant 0. \tag{3.33}$$

If in this expression we choose f identically equal to one then we get

$$\int_{\partial Q} H \, \mathrm{d}\sigma \geqslant \int_{\partial Q} |\underline{b}|^2 \mathrm{d}\sigma. \tag{3.34}$$

By the Gauss-Bonnet theorem (see e.g. [28]) $\int H \, \mathrm{d}\sigma = 4\pi\chi(\partial Q)$ (notice that H is twice the Gaussian curvature of $(\partial Q, \underline{h})$), where $\chi(\partial Q)$ is the Euler characteristic of ∂Q [$\chi(\partial Q)$ is 2 for the sphere, zero for the torus, and negative for any other orientable ∂Q]. Hence

$$4\pi\chi(\partial Q) \geqslant \int_{\partial Q} |\underline{b}|^2 \mathrm{d}\sigma. \tag{3.35}$$

As long as $\underline{b} \neq \underline{0}$ (i.e. $(\partial Q, \underline{h})$ has some shear) (3.35) implies that a ∂Q for which (3.26) holds true and gives rise to a stable minimal manifold in the physical space $(S, \underline{g})$ is topologically a two-sphere. [This fact could have been anticipated by recalling a well-known result by Hawking, [29], according to which an apparent horizon must be a two-sphere provided that the dominant energy condition holds true]. It is worth noticing that if $\underline{b} = \underline{0}$, then (3.35) may be also consistent with ∂Q being topologically a two-torus. However we can dismiss such an eventuallity as non generic by noticing that in this case (e.g. see [23]) such torus is embedded as a flat, totally geodesic two-torus in a flat (topologically non-trivial) $(S, \underline{g})$.

A further interesting property that a ∂Q, for which (3.26) holds true, must satisfy is also strictly related to (3.35). It comes about when we express the integral $\int |\underline{b}|^2 \mathrm{d}\sigma$ appearing in (3.35) in terms of $\underline{\hat{h}}$ and $\underline{\hat{b}}$, the first and the second fundamental forms of the embedding of ∂Q in the base space $(S, \underline{\hat{g}})$. A straightforward calculation yields

$$b_{ik} = \varphi^2(\hat{b}_{ik} - 2\hat{h}_{ik}\hat{s}^j\hat{\nabla}_j W), \qquad h_{ik} = \varphi^4 \hat{h}_{ik}.$$

Hence, if we take into account (3.26)

$$|\underline{b}|^2 = \varphi^{-4}\left[|\underline{\hat{b}}|^2 - \frac{1}{2}(\mathrm{tr}\,(\underline{\hat{b}}))^2\right], \qquad \mathrm{d}\sigma = \varphi^4 \mathrm{d}\hat{\sigma}$$

and (3.35) reduces to

$$4\pi\chi(\partial Q) \geqslant \int_{\partial Q} \left(|\underline{\hat{b}}|^2 - \frac{1}{2}(\mathrm{tr}\,(\underline{\hat{b}}))^2\right) \mathrm{d}\hat{\sigma}. \tag{3.36}$$

The integrand in (3.36) is just square shear that $(\partial Q, \underline{\hat{h}})$ inherits from its embedding in $(S, \underline{\hat{g}})$. Hence, the two-surfaces for which (3.26) may hold true (in the stable sense specified above), can be found only among the two-spheres that satisfy the low-shear condition

$$\int_{\partial Q} \left(|\underline{\hat{b}}|^2 - \frac{1}{2}(\mathrm{tr}\,(\underline{\hat{b}}))^2\right) \mathrm{d}\hat{\sigma} \leqslant 8\pi. \tag{3.37}$$

The development of apparent horizons

Let us now discuss the actual possibility that a two-surface $(\partial Q, \hat{\underline{h}})$ for which (3.26) holds true develops. To this end let $(\partial\Sigma, \hat{\underline{h}})$ be a two-surface in $(S, \hat{\underline{g}})$. According to the above considerations we assume that $\partial\Sigma$ is topologically a 2-sphere, and that its mean curvature with respect to the outward normal is, on the average, negative. Namely

$$\int_{\partial\Sigma} (\mathrm{tr}\,(\hat{\underline{b}}))\, \mathrm{d}\hat{\sigma} < 0, \tag{3.38}$$

(notice that according to our conventions, the sign of the mean curvature is chosen so that the sphere in the Euclidean space has negative mean curvature with respect to the outward normal). Let us denote by $\mathrm{Emb}\,(\partial\Sigma, S, \hat{\underline{g}})$ the set of embeddings $i : \partial\Sigma \simeq \mathbb{S}^2 \to (S, \hat{\underline{g}})$ such that $i(\partial\Sigma) \subset S$ satisfies condition (3.34), and let us assume that i as an mapping belongs to the Sobolev space of maps $H^q_p(\partial\Sigma, S)$. Consider, for a given solution W of problem (3.22) on $(S, \hat{\underline{g}})$, the following mapping from $\mathrm{Emb}\,(\partial\Sigma, S, \hat{\underline{g}})$ to the smooth functions on $\partial\Sigma$:

$$F(\hat{\underline{g}}, \partial\Sigma) \equiv -(\mathrm{tr}\,(\hat{\underline{b}}))\,|_{\partial\Sigma} + 4(\hat{\underline{s}} \circ \hat{\nabla} W)\,|_{\partial\Sigma}. \tag{3.39}$$

The zeros of this mapping, if any, characterize the embeddings $i : \mathbb{S}^2 \to (S, \hat{\underline{g}})$ for which (3.26) holds true.

With these premises let us apply Gauss's theorem to equation (3.22) over the (simply connected) region Σ bounded by $\partial\Sigma$. In this way we get

$$\int_{\partial\Sigma} (\hat{\underline{s}} \circ \hat{\nabla} W)\, \mathrm{d}\hat{\sigma} = \int_{\Sigma} \left(\frac{1}{8}\, R(\hat{\underline{g}}) - |\hat{\nabla} W|^2\right) \mathrm{d}v_{\hat{\underline{g}}},$$

which, on taking into account the definition of F, yields

$$\int_{\partial\Sigma} F(\hat{\underline{g}}, \partial\Sigma)\,\mathrm{d}\hat{\sigma} \leqslant -\int_{\partial\Sigma} (\mathrm{tr}\,(\hat{\underline{b}}))\,\mathrm{d}\hat{\sigma} + \frac{1}{2}\int_{\Sigma} R(\hat{\underline{g}})\,\mathrm{d}v_{\hat{\underline{g}}}. \tag{3.40}$$

This provides an upper bound to the average value of F over $\partial\Sigma$ in terms of the average mean curvature of $i(\partial\Sigma)$ and of the integrated scalar curvature $R(\hat{\underline{g}})$ in the region Σ. Notice in particular that in (3.40) there is no more reference to the actual value W of the solution of (3.22), which instead enters directly the definition of F.

Let U denote an open subset of S. Consider now a compactly supported deformation $\beta \to \hat{g}(\beta)$, $0 \leqslant \beta \leqslant 1$, of the given base metric $\hat{g}$, which holds $(S, \hat{g})$ outside a compact subset of U fixed. Let $\{i_\rho(\partial\Sigma)\}$, $(\rho = 0, 1, 2, \ldots)$, denote a sequnece of embeddings belonging to $\mathrm{Emb}\,(\partial\Sigma, S, \hat{g})$ such that $U \subseteq \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$ (Σ_ρ denoting the domain bounded by $i_\rho(\partial\Sigma)$). We choose the deformation $\beta \to \to \hat{g}(\beta)$ so as to make $R(\hat{g}(\beta))$ more and more negative in U. As follows from (3.40) even if initially $\int F \mathrm{d}\hat{\sigma} > 0$ (as can be always arranged) $\int F \mathrm{d}\hat{\sigma}$ decreases as the deformation goes on and, provided that the negative lower bound of $R(\hat{g}(\beta))$ gets sufficiently large, we can make $\int F \mathrm{d}\hat{\sigma}$ vanish on some $i_\rho(\partial\Sigma)$, let us say on $i^*(\partial\Sigma)$. If $\| F(i^*(\partial\Sigma)) \|_{H^g_\rho}$ is sufficiently small, then we expect that in the same isotopy class $I(i^*(\partial\Sigma))$ of $i^*(\partial\Sigma)$ there exists an element $\bar{i}(\partial\Sigma)$ for which $F(\bar{i}(\partial\Sigma)) = 0$ holds pointwise [$I(i^*(\partial\Sigma))$ denotes the collection of embeddings $i_t(\partial\Sigma)$ which can be expressed as $i_t(\partial\Sigma) = \zeta_t(i^*(\partial\Sigma))$, $|t| < 1$, with $\zeta_t : (-1, 1) \times S \to S$, a smooth diffeomorphism for each t, defined by $\zeta_t(x) = \zeta(t, x)$].

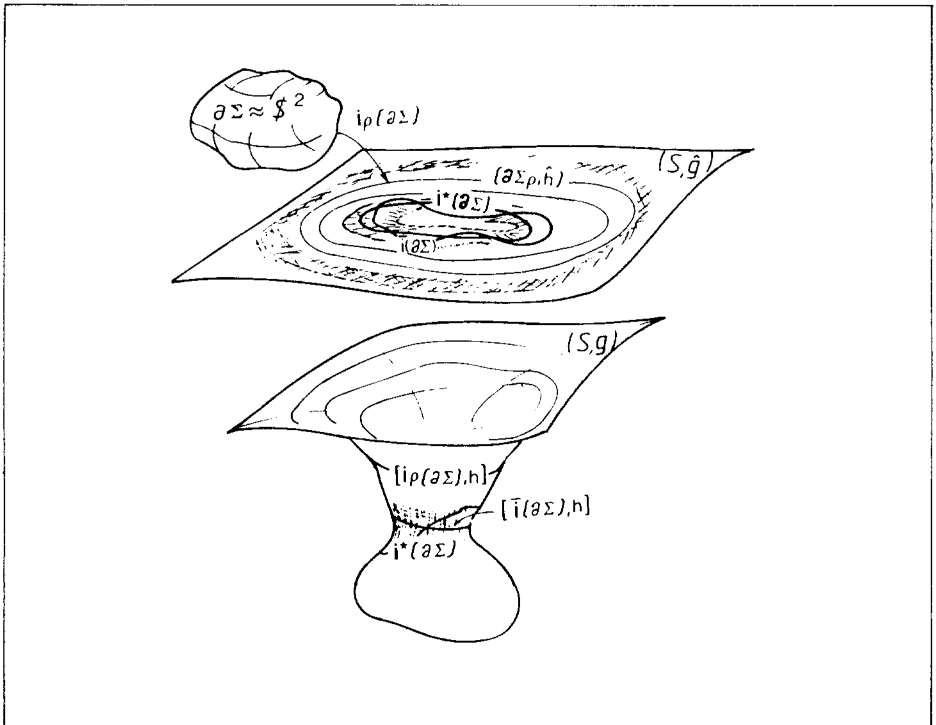

Figure 6. The development of an apparent horizon in the physical space.

To make this remark more precise we apply the inverse function theorem to F. Begin by using the field of geodesics orthogonal to $i^*(\partial\Sigma)$ to parametrize a neighbourhood of $i^*(\partial\Sigma)$ by $|i^*(\partial\Sigma)|\times(-\epsilon,\epsilon)$ so that $(x,0)$ corresponds to $x\in i^*(\partial\Sigma)$ and so that $(x)\times(-\epsilon,\epsilon)$ is the geodesic segment orthogonal to $i^*(\partial\Sigma)$ at x, parametrized by arc length. Now let us consider the formal linearization of $F(\underline{\hat{g}}, i^*_\epsilon(\partial\Sigma))$ around $i^*(\partial\Sigma)$. On utilizing the isotopy induced by the above parametrization and (3.28) we get for any smooth deformation $\underline{\hat{z}} = f\underline{\hat{s}}$ around $i^*(\partial\Sigma)$

$$DF(i^*_\epsilon(\partial\Sigma), \underline{\hat{g}})|_{\epsilon=0}\circ f = -(|\underline{\hat{b}}|^2 + \hat{\mathrm{R}}\mathrm{ic}\,(\underline{\hat{s}},\underline{\hat{s}}) - 4\,\hat{\mathrm{H}}\mathrm{ess}\, W(\underline{\hat{s}},\underline{\hat{s}}) + {}^{(2)}\hat{\Delta})f, \tag{3.41}$$

where f is a smooth function with compact support on $i^*(\partial\Sigma)$, ${}^{(2)}\Delta$ is the Laplace-Beltrami operator on $(\partial\Sigma, \underline{\hat{h}})$ and where Hess $\hat{W}$ is the Hessian of W in $(S, \underline{\hat{g}})$ (locally [Hess $W(\underline{\hat{s}},\underline{\hat{s}})] = \hat{s}^i\hat{s}^k\hat{\nabla}_i\hat{\nabla}_k W$).

From (3.41) we see that the linear mapping $f\to DF\circ f$ is an isomorphism between open sets of scalar functions in $H^q_p(i^*(\partial\Sigma))$ if (see e.g. [12], Th. (4.18))

$$[|\underline{\hat{b}}|^2 + \hat{\mathrm{R}}\mathrm{ic}\,(\underline{\hat{s}},\underline{\hat{s}}) - 4\,\mathrm{Hess}\, W(\underline{\hat{s}},\underline{\hat{s}})]|_{i^*(\partial\Sigma)} < 0. \tag{3.42}$$

If $i^*(\partial\Sigma)$ is such that (3.42) holds true, then by the inverse function theorem $F(i(\partial\Sigma))$ is a local homeomorphism of a neighbourhood $A(i^*(\partial\Sigma))\subset \mathrm{Emb}\,(\partial\Sigma, S, \underline{\hat{g}})$ of $i^*(\partial\Sigma)$ to a neighbourhood of $F(i^*(\partial\Sigma))$. In particular if $\|F(i^*(\partial\Sigma))\|_{H^q_p}$ is sufficiently small, then the sequence of embeddings

$$i_{m+1}(\partial\Sigma) = i_m(\partial\Sigma) - |DF(i(\partial\Sigma))|^{-1}|_{i^*(\partial\Sigma)}\cdot F(i_m(\partial\Sigma)) \tag{3.43}$$

converges to the unique embedding $i(\partial\Sigma)$ in $A(i^*(\partial\Sigma))$ such that $F(i(\partial\Sigma)) = 0$. Notice that condition (3.42), if satisfied, besides implying the solvability of (3.26) also implies that the embedding $i(\partial\Sigma)$ generated according the above procedure actually minimizes (Vol $(i(\partial\Sigma), \underline{h} = \varphi^4\underline{\hat{h}})$ among all competing embeddings.

To see this let $\underline{\hat{z}} = f\underline{\hat{s}}$, $f\in C^\infty(\partial\Sigma)$, denote a variation vector field associated with the geodesic parametrization of $i^*(\partial\Sigma)\times(-\epsilon,\epsilon)$ introduced above. Let

$$\mathrm{Vol}\,[i(\partial\Sigma), \underline{h} = \varphi^4\underline{\hat{h}}] = \int_{i(\partial\Sigma)} \varphi^4\,\mathrm{d}\hat{\sigma}$$

be the area functional associated with $i(\partial\Sigma)$ in the physical metric $\underline{g} = \varphi^4\underline{\hat{g}}$, but thought as a functional of the embedding of $\partial\Sigma$ in the given base space $(S, \underline{\hat{g}})$. Consider now the first and the second variation of this area functional under the action of $\underline{\hat{z}}$, around an embedding $\bar{i}(\partial\Sigma)$ for which $F(\bar{i}(\partial\Sigma)) = 0$. An easy computation (as for (3.27)) provides

$$\frac{\mathrm{d}}{\mathrm{d}\hat{z}} \mathrm{Vol}\,[i(\partial\Sigma), \underline{h} = \varphi^4 \hat{\underline{h}}]\,|_{\bar{i}(\partial\Sigma)} = \int_{\bar{i}(\partial\Sigma)} \varphi^4 f F(\bar{i}(\partial\Sigma))\,\mathrm{d}\hat{\sigma} = 0,$$

(3.44)
$$\frac{\mathrm{d}^2}{\mathrm{d}\hat{z}^2} \mathrm{Vol}\,[i(\partial\Sigma), \underline{h} = \varphi^4 \hat{\underline{h}}]\,|_{\bar{i}(\partial\Sigma)} = -\int_{\bar{i}(\partial\Sigma)} \varphi^4 f\,{}^{(2)}\hat{\Delta} f\,\mathrm{d}\hat{\sigma} - \int_{\bar{i}(\partial\Sigma)} [\,|\hat{\underline{b}}|^2 + \hat{\mathrm{R}}\mathrm{ic}\,(\hat{\underline{s}}, \hat{\underline{s}}) - 4\,\hat{\mathrm{H}}\mathrm{ess}\,(\hat{\underline{s}}, \hat{\underline{s}})] f^2 \varphi^4\,\mathrm{d}\hat{\sigma}.$$

The sign of the first integral in (3.44) is not so easily estimated as in the analogous expression (3.27), since we cannot integrate by parts owing to the presence of the factor φ^4. However we can proceed as follows. Let us suppose that

$$\int_{\bar{i}(\partial\Sigma)} \varphi^4 f\,{}^{(2)}\hat{\Delta} f\,\mathrm{d}\hat{\sigma} > 0.$$

This implies that for some open subdomain $U \subset \partial\Sigma$

$$f\,{}^{(2)}\hat{\Delta} f = \frac{1}{2}\,{}^{(2)}\hat{\Delta} f^2 - |\hat{\nabla} f|^2 > 0, \qquad f|_{\partial U} = 0.$$

That is ${}^{(2)}\hat{\Delta} f^2 > 0$ in U, $f^2|_{\partial U} = 0$, which by Hopf's maximum principle [12] implies $f = 0$ on U. Thus we get a contradiction and

$$\int_{\bar{i}(\partial\Sigma)} \varphi^4 f\,{}^{(2)}\hat{\Delta} f\,\mathrm{d}\hat{\sigma} \leqslant 0.$$

Introducing this into (3.44) and taking into account (3.42) we get that corresponding to the embedding $\bar{i}(\partial\Sigma)$

$$\frac{\mathrm{d}^2}{\mathrm{d}\hat{z}^2} \mathrm{Vol}\,[i(\partial\Sigma), \underline{h} = \varphi^4 \hat{\underline{h}}]\,|_{\bar{i}(\partial\Sigma)} \geqslant 0$$

as claimed.

Once clarified the geometrical meaning of condition (3.42), it remains to check if this condition can be met. To this end, let $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{s}})$ be an orthonormal triad with $(\hat{\underline{e}}_1, \hat{\underline{e}}_2)$ unit vector fields tangent to the embedding $i(\partial\Sigma) \subset (S, \underline{g})$. On taking into account equation (3.22) we can write

$$\hat{\Delta} W = \hat{\text{H}}\text{ess } W(\hat{\underline{s}},\hat{\underline{s}}) + \hat{\text{H}}\text{ess } W(\hat{\underline{e}}_1,\hat{\underline{e}}_1) + \hat{\text{H}}\text{ess } W(\hat{\underline{e}}_2,\hat{\underline{e}}_2) = \tag{3.45}$$
$$= \frac{1}{8} R(\hat{\underline{g}}) - |\hat{\nabla} W|^2 .$$

But

$$\hat{\text{H}}\text{ess } W(\hat{\underline{e}}_1,\hat{\underline{e}}_1) + \hat{\text{H}}\text{ess } W(\hat{\underline{e}}_2,\hat{\underline{e}}_2) = {}^{(2)}\hat{\Delta} W - (\hat{\underline{s}}^r \hat{\nabla}_r W)(\text{tr }(\hat{\underline{b}})).$$

Hence, after rearranging terms in (3.45), we get

$$R(\hat{\underline{g}}) - 8\, \hat{\text{H}}\text{ess } W(\hat{\underline{s}},\hat{\underline{s}}) = 8\,|\hat{\nabla} W|^2 + 8\,{}^{(2)}\hat{\Delta} W - 8(\hat{s}^r \hat{\nabla}_r W)(\text{tr }(\hat{\underline{b}})). \tag{3.46}$$

On the other hand condition (3.42) can be rewritten as (by using the Gauss curvature equation as in (3.31)):

$$[\,|\hat{\underline{b}}|^2 + (\text{tr }(\hat{\underline{b}}))^2 - \hat{H} + R(\hat{\underline{g}}) - 8\,\hat{\text{H}}\text{ess } W(\hat{\underline{s}},\hat{\underline{s}})]\,|_{i(\partial\Sigma)} < 0, \tag{3.47}$$

where $\hat{H}$ is the scalar curvature of $(i(\partial\Sigma), \hat{\underline{h}})$. If we rewrite the term $(\hat{s}^r \hat{\nabla}_r W)$ $(\text{tr }(\hat{\underline{b}}))$, appearing in (3.46), as $8(\hat{s}^r \hat{\nabla}_r W) = 2F + 2(\text{tr }(\hat{\underline{b}}))$ (by using (3.39)), and introduce the resulting expression for $|R(\hat{\underline{g}}) - 8\,\hat{\text{H}}\text{ess } W(\hat{\underline{s}},\hat{\underline{s}})|$ in (3.47), we get that (3.42) is equivalent to requiring

$$[\,|\hat{\underline{b}}|^2 - \hat{H} + 8\,|\hat{\nabla} W|^2 + 8\,{}^{(2)}\hat{\Delta} W - 2(\text{tr }(\hat{\underline{b}}))F - (\text{tr }(\hat{\underline{b}}))^2]\,|_{i(\partial\Sigma)} < 0. \tag{3.48}$$

But

$$8\,|\hat{\nabla} W|^2 = 8\,|{}^{(2)}\hat{\nabla} W|^2 + 8(\hat{s}^r \hat{\nabla}_r W)^2 = \frac{1}{2} F^2 + \frac{1}{2} (\text{tr }(\hat{\underline{b}}))^2 + 8\,|{}^{(2)}\hat{\nabla} W|^2$$

where we have used again (3.39). If we introduce the shear tensor $\hat{\underline{a}} = \hat{\underline{b}} - \frac{1}{2} \hat{\underline{h}}(\text{tr }(\hat{\underline{b}}))$ associated with $\underline{b}$, and plug in these latter expressions in (3.48) we finally get that (3.42) can be equivalently rewritten as:

$$\left[|\hat{\underline{a}}|^2 - \hat{H} + 8|{}^{(2)}\hat{\nabla} W|^2 + 8\,{}^{(2)}\hat{\Delta} W + \frac{1}{2} F^2 - 2(\text{tr }(\hat{\underline{b}}))F\right]\Bigg|_{i(\partial\Sigma)} < 0. \tag{3.49}$$

In this form condition (3.42) suggests itself for which two-surfaces look for in order to get it satisfied. For, if we fix our attention on the equipotential surfaces $W = \text{Cost.}$, (3.49) reduces to

$$\left[|\hat{\underline{a}}|^2 - \hat{H} + \frac{1}{2} F^2 - 2(\text{tr }(\hat{\underline{b}}))F\right]\Bigg|_{i(\partial\Sigma)} < 0. \tag{3.50}$$

And, as soon as $F(i(\partial\Sigma))$ becomes sufficiently small this is satisfied provided that the shear $|\hat{\underline{a}}|$ is not too large [notice that, as a consequence of (3.37), $|\hat{\underline{a}}|$ on the average is smaller than $\hat{H}$].

More in general, if we want to check the feasibility of condition (3.42) on two surfaces which are distinct from the equipotentials $W = \text{Const.}$, (3.49) *shows that this condition cannot be satisfied corresponding to a generic wild embedding of a* $\partial\Sigma$ *in* $(S, \hat{\underline{g}})$*. And even if we assume a low shear condition (such as (3.37), for instance), so to have* $(i(\partial\Sigma), \hat{\underline{h}})$ *sufficiently round in* $(S, \hat{\underline{g}})$ *it is further necessary that* $i(\partial\Sigma)$ *is sufficiently «near» an equipotential* $W = \text{Const.}$*. Namely, we must require that* $|{}^{(2)}\hat{\nabla} W|_{i(\partial\Sigma)}$ *is small enough*. The tolerance on $|{}^{(2)}\hat{\nabla} W|$ can be easily estimated on assuming that correspondign to $i(\partial\Sigma)$, $F(i(\partial\Sigma))$ is a small positive constant, say ϵ, and integrating the resulting expression (3.49) over $i(\partial\Sigma)$. In this way we get

$$\int_{i(\partial\Sigma)} |\hat{\underline{a}}|^2 \mathrm{d}\hat{\sigma} + 8 \int_{i(\partial\Sigma)} |{}^{(2)}\hat{\nabla} W|^2 \mathrm{d}\hat{\sigma} + \frac{1}{2}\epsilon^2 \int_{i(\partial\Sigma)} \mathrm{d}\hat{\sigma} - \tag{3.51}$$
$$- 2\epsilon \int_{i(\partial\Sigma)} (\mathrm{tr}\,(\hat{\underline{b}}))\mathrm{d}\hat{\sigma} < \int_{i(\partial\Sigma)} \hat{H} \mathrm{d}\hat{\sigma}.$$

Since $i(\partial\Sigma)$ is topologically a two-sphere (3.51) implies (on applying the Gauss--Bonnet theorem)

$$\int_{i(\partial\Sigma)} |{}^{(2)} VW|^2 \mathrm{d}\hat{\sigma} < \pi + \frac{1}{4}\epsilon \int_{i(\partial\Sigma)} (\mathrm{tr}\,(\hat{\underline{b}}))\mathrm{d}\hat{\sigma}.$$

An explicit upper bound for $|{}^{(2)}\hat{\nabla} W|$ in terms of the geometry of the given base space $(S, \hat{\underline{g}})$ can be provided as follows. Let $I(i(\partial\Sigma)) = i(\partial\Sigma) \times (-\epsilon, \epsilon)$ denote a tubular neighbourhood of the given $i(\partial\Sigma)$. Let $(B_i(r_i))$ be a finite covering of $I(i(\partial\Sigma))$ by geodesic balls $B_i(r_i) \subset (S, \hat{\underline{g}})$ of radius r_i. For each $B_i(r_i)$, $0 < \beta < 1$, $p \geqslant 3$, there is, as already recalled (see p. 164), an L^p bound for $|\hat{\nabla} W|^2$ of the form:

$$\left[\int_{B((1-\beta)r_i)} |\hat{\nabla} W|^{2p} \mathrm{d}v_{\hat{\underline{g}}}\right]^{1/p} \leqslant a\left(\int_{B_i(r_i)} K^p \mathrm{d}v_{\hat{\underline{g}}}\right)^{1/p} + \tag{3.52}$$
$$+ \frac{1}{8} a p^{1/2} \left(\int_{B_i(r_i)} (R(\hat{\underline{g}}))^p \mathrm{d}v_{\hat{\underline{g}}}\right)^{1/p} + a\beta^{-2} p r_i^{-2} [\mathrm{Vol}\,(B_i(r_i), \hat{\underline{g}})]^{1/p}.$$

Since $|\hat{\nabla} W|^2 = |{}^{(2)}\hat{\nabla} W|^2 + |\hat{s}^r \hat{\nabla}_r W|^2 \geqslant |{}^{(2)}\hat{\nabla} W|^2$, this yields

$$\left[\int_{B_i((1-\beta)r_i)} |{}^{(2)}\hat{\nabla} W|^{2p} \, \mathrm{d}v_{\underline{\hat{g}}}\right]^{1/p} \leqslant C_i, \tag{3.53}$$

where C_i is a shorthand for the expression appearing at the right side of (3.52). Notice that by choosing suitably r_i and β in C_i, (3.53) allows us to estimate to a great accuracy $|{}^{(2)}\hat{\nabla} W|$ on $B_i((1-\beta)r_i) \cap i(\partial\Sigma)$ and hence to control the size of $|{}^{(2)}\hat{\nabla} W|$ on $i(\partial\Sigma)$ in terms of the geometry of the ambient space $(S, \underline{\hat{g}})$ in a neighbourhood of $i(\partial\Sigma)$' Finally notice that we can also control the size of ${}^{(2)}\hat{\Delta} W$ appearing in (3.49). Let $U \subset i(\partial\Sigma)$ a domain in $i(\partial\Sigma)$. For any sufficiently smooth functions on $i(\partial\Sigma)$, f, ζ, with ζ having compact support in U, we can write

$$\int_{i(\partial\Sigma)} f^2 |{}^{(2)}\hat{\nabla}\zeta|^2 \, \mathrm{d}\hat{\sigma} \geqslant \int_{i(\partial\Sigma)} \zeta^2 f \, {}^{(2)}\hat{\Delta} f \, \mathrm{d}\hat{\sigma}. \tag{3.54}$$

Let choose f so to have $f = \exp(W)$. In this case (3.54) reduces to

$$\int_{i(\partial\Sigma)} f^2 |{}^{(2)}\hat{\nabla}\zeta|^2 \, \mathrm{d}\hat{\sigma} \geqslant \int_{i(\partial\Sigma)} \zeta^2 f^2 [{}^{(2)}\hat{\Delta} W + |{}^{(2)}\hat{\nabla} W|^2] \, \mathrm{d}\hat{\sigma}.$$

Since $\sup f|_{i(\partial\Sigma)} \equiv q < \infty$, this implies

$$\int_{i(\partial\Sigma)} |{}^{(2)}\hat{\nabla}|^2 \, \mathrm{d}\hat{\sigma} \geqslant \int_{i(\partial\Sigma)} \zeta^2 \left\{\frac{f^2}{q} [{}^{(2)}\hat{\Delta} W + |{}^{(2)}\hat{\nabla} W|^2]\right\} \mathrm{d}\hat{\sigma}. \tag{3.55}$$

Now let us remark that the above expression is a conformal invariant [this immediately follows by noticing that if $\tilde{h}_{ij} = \alpha^4 \hat{h}_{ij}$ then ${}^{(2)}\tilde{\Delta} f = \alpha^{-4} {}^{(2)}\hat{\Delta} f$, $|{}^{(2)}\nabla f|^2 = \alpha^{-4} |{}^{(2)}\hat{\nabla} f|^2$]. Thus, modulo a one-point compactification and a diffeomorphism, we can discuss (3.55) on the Euclidean plane. There, the validity of the variational inequality (3.55) for every smooth ζ with compact support, implies that

$$\int_D \frac{f^2}{q} [{}^{(2)}\Delta W + |{}^{(2)}\nabla W|^2] \, \mathrm{d}^2 x \leqslant C, \tag{3.56}$$

C being a constant that can be estimated, and where D is the union of the domains in $\mathbb{R}^2$ where the integrand is positive. Clearly by conformal invariance (3.56) holds true also on the original two-manifold $i(\partial\Sigma)$ providing there an upper bound for the positive contribution of ${}^{(2)}\hat{\Delta}W$.

Estimates (3.53) and (3.56) imply that in any case, regardless of the use of equipotentials, condition (3.42) can be satisfied corresponding to embeddings $i(\partial\Sigma)$ which probe regions in $(S,\underline{\hat{g}})$ of sufficiently small curvature. Since (3.53) and (3.56) only depend on the local properties of $(S,\underline{\hat{g}})$ nearby $i(\partial\Sigma)$, the curvature in the domain Σ bounded by $i(\partial\Sigma)$ needs not to be weak. In particular the scalar curvature $R(\underline{\hat{g}})$ within can be made large and negative. Correspondingly $f(i(\partial\Sigma))$ gets smaller and smaller and the procedure delineated above, via the inverse function theorem, can be actually implemented, yielding a unique embedding $\bar{i}(\partial\Sigma)$, in the same isotopy class of $i(\partial\Sigma)$, for which $|\hat{\nabla}_i\hat{s}^i + 4(\hat{s}^j\hat{\nabla}_j \log\varphi)|_{\bar{i}(\partial\Sigma)} = 0$, and which realizes an apparent horzon in the physical space.
According to the above results we can now complete the picture of «topological bifurcation» occurring for the physical data as the given base space is deformed in such a way as to develop larger and larger negative scalar curvature. As this happens; and $\lambda_1(B,\underline{\hat{g}})$ goes through zero, singular solutions, $(S_C,\underline{g})$ and $(S_0,\underline{g})$ of the Hamiltonian constraint develop. From the above analysis we cannot conclude that apparent horizons form in $(S_C,\underline{g})$ and $(S_0,\underline{g})$ as soon as $\lambda_1(B,\underline{\hat{g}}) \to 0$. However, as the negative part of the scalar curvature increases further, either in $(S_0,\underline{g})$ or in $(S_C,\underline{g})$ apparent horizons disconnecting the singular regions from observation do form. [It is clear that even if the above discussion of the development of minimal two-surface was referring to the asymptotically euclidean singular solution $(S_0,\underline{g})$, its conclusions can be extended also to $(S_C,\underline{g})$ without substantial alterations]. The open singular solution $(S_0,\underline{g})$ eventually provides data for a time--symmetric black-hole configuration the (ADM)mass of which is well-defined and positive [this last statement follows from the positive mass theorem (see e.g. [30] for a recent discussion); in particular from the positive mass theorem for time-symmetric black-hole spacetimes proved by P.S. Jang [31]]. Similarly, the other singular solution $(S_C,\underline{g})$ eventually ends up in describing a perfectly well--behaved data set corresponding to a closed physical space containing a black hole in a momentarily static configuration.

Thus, in general, corresponding to a curve of base metrics $\beta \to \underline{\hat{g}}(\beta)$ for which $\lambda_1(B,\underline{\hat{g}}(\beta))$ decreases, goes through zero, and attains negative values, there corresponds a family of physical metrics $\beta \to \underline{g}(\beta)$, solutions of the Hamiltonian constraint which starts as an asymptotically euclidean vacuum, time-symmetric, initial data set ($\lambda_1 > 0$), then bifurcates ($\lambda_1 = 0$) into two distinct curves of singular solutions $(S_0,\underline{g})$ and $(S_C,\underline{g})$ modelled on topologically non-trivial manifolds, and finally settles down ($\lambda_1 < 0$) to regular black-holds data sets. [As λ_1 further decreases we may suspect that this process goes on ad infinitum, with

S_0 and S_C themselves bifurcating and giving rise to more and more complicate configurations of possible singular and well-behaved time-symmetric data. Notice that Cantor and Piran's heuristic model suggests this eventuality].

In appears likely that the above mechanism carries over to the most general (i.e., non-vacuum, non-time-symmetric case). However no proof of this fact is available [a general proof of the occurrence of apparent horizons, in presence of matter, and for regions of «large radius», for asymptotically Euclidean initial data sets has been provided by Schoen and Yau [32]. However, their, proof, which is of an indirect nature, cannot be easily accomodated within the conformal approach]. Finally, it is clear that the above topological bifurcation mechanism is, within the classical theory, just a pleasant characteristic of the formalism, since, as already stressed, topological changes in the spatial sections are dynamically suppressed by the evolutive part of Eistein's equations (1). The real issue raised by the considerations of this paragraph will rather concern the relevance of this bifurcation mechanism at a quantum level.

4. THE TOPOLOGY OF CLOSED PHYSICAL SPACE

When the three-manifold S on which physical space is assumed to be modelled is closed (i.e. compact and without boundary) the interaction between the topology and the energetic content of the physical space appears on a different footing than in the asymptotically euclidean case. These differences, however, are only apparent and eventually we end up with deep connections between the two cases.

Let us start by noticing that now solvability conditions for the Lichnerowicz-York equation (2.8) are not of much help as they are in the open case. For S closed, problem (2.8) has been discussed by Choquet-Bruhat [33] and by O'Murchadha and York [34], assuming that S can be embedded in the final spacetime so to have constant mean extrinsic curvature (i.e. k constant). They proved that, for each given set of York data $(\underline{\hat{g}}, \underline{\hat{A}}_{\perp}, \hat{\mu}, \hat{J}, k)$, (2.8) admits a unique bounded solution except for a set of York data corresponding to unphysical configuration of matter and gravitational radiation (e.g. vacuum time-symmetric data on closed three-manifolds). In particular, from their analysis it follows that no curvature restriction (say, such as (3.4)) plays any significant role in discussing existence, uniqueness and stability of solutions of (2.8), whenever some matter or gravitational radiation is present. This circumstance raises the question of which topological information may be provided by the Hamiltonian constraint

(1) To avoid any misunderstanding notice that the topology changes discussed is a change along a family of initial data, but not a change in time.

when S is closed.

We can try to get topological information by using known topological obstructions to particular values of the sign of the scalar curvature $R(\underline{g})$ appearing in the Hamiltonian constraint (2.5). Clearly, this line of approach must be taken cautiously. For, while initial data sets are given in conformal equivalence classes, the scalar curvature is not a conformal invariant, changing, in general, either its sign or its support under a conformal transformation. On the other hand, such an approach has a strong physical appeal. For, a rewriting of the Hamiltonian constraint (2.5) as

$$R(\underline{g}) = 16\pi\mu + \underline{\tilde{K}} \circ \underline{\tilde{K}} - \frac{2}{3}\, k^2 \tag{4.1}$$

tells us that the three-manifold $(S, \underline{g})$, modelling the physical space, tends to support those riemannian structures which give rise to not «too much» negative, or even positive, scalar curvature, if enough matter and gravitational radiation are present. Actually, the only negative contribution to $R(\underline{g})$ comes from a kinematical term: $(k)^2$. Even if we cannot simply dispose of this term by referring to its non-dynamical nature (e.g., $k = 0$, for S closed, cannot be assumed in general without strongly restricting the gravitational configuration under study), we may reasonably expect that $\frac{2}{3}\,(k)^2$ is dominated, in physically realistic situations, by the «matter plus radiation» terms μ and $\underline{\tilde{K}} \circ \underline{\tilde{K}}$. If this dominance were to hold everywhere, then we would be in a very nice situation. For, there is a number of topological obstructions (see below) that must be necessarily satisfied by closed Riemannian manifolds supporting positive scalar curvature. In general, we can only assume that there are regions in which $\left(16\pi\mu + \underline{\tilde{K}} \circ \underline{\tilde{K}} - \frac{2}{3}\,(k)^2\right) > 0$ holds, and regions where it does not (e.g. those regions where matter and radiation are absent). This case cannot directly yield the same information of the former case. For, as is known [35], [36], a negative scalar curvature does not impose particular restrictions on the topology of the underlying manifold S. However, since initial data sets are defined in conformal equivalence classes, it may happen, if $(k)^2$ is not too large, that in the same conformal equivalnce class defined by the physical metric $\underline{g}$ there are representative elements $\underline{g}'$ (in general, different from the given base metric $\underline{\hat{g}}$) whose associated scalar curvature is pointwise positive. This circumstance will reconduce us to former case (for a smooth conformal trasformation does not alter either the topology of S or the given initial set) and will allow us to get nontrivial topological information from the Hamiltonian constraint.

With these premises, let us consider on $(S, \underline{g})$ the functional

$$Y(g, f) \equiv \left(\int_S f^6 \mathrm{d}v_{\underline{g}}\right)^{-1/3} \left(8 \int_S |\nabla f|^2 \mathrm{d}v_{\underline{g}} + \int_S R(\underline{g}) f^2 \mathrm{d}v_{\underline{g}}\right), \tag{4.2}$$

with $f \in H_1^2(S)$, $f > 0$. This functional was introduced by Yamabe [37], in discussing the well-known conjecture named after him: in each conformal equivalence class $\{\underline{g}\}$ over a simply connected closed riemannian manifold $(V^n, \underline{g})$ $(n \geqslant 3)$, there are representative elements $\underline{g}'$ (not necessarily unique) the associated scalar curvature of which $R(\underline{g}')$ is a constant.

Our interest in (4.2) rests on the observation (due to Aubin [12], p. 126) that the numerical quantity

$$I(\underline{g}) = \operatorname*{Inf}_{f \in H_1^2(S),\ f > 0} Y(\underline{g}, f) \tag{4.3}$$

if attained, is a conformal invariant. That is, it only depends on the conformal structure associated with the Riemannian manifold $(S, \underline{g})$. In particular, if $\hat{\underline{g}}$ is the base metric associated with a given York data set $(\hat{\underline{g}}, \hat{\underline{A}}_\perp, \hat{\mu}, \hat{\underline{J}}, k)$ and $\underline{g}$ is the physical metric obtained by solving the Lichnerowicz-York equation (2.8), then we must necessarily have

$$I(\hat{\underline{g}}) = I(\underline{g}). \tag{4.4}$$

There reader will have noticed the similarity between (4.1), (4.2) and the variational characterization (3.6) of the first eigenvalue of the first eigenvalue of the conformal Laplacian for a domain B. Here, however, it is much more difficult to prove that the infimum (4.3) is actually attained. It can be easily shown (by using Lichnerowicz' formula (3.8)) that this infimum is attained by the solution of the second order, non-linear, elliptic problem (the Yamabe problem)

$$-8\Delta\varphi + R(\underline{g})\varphi = R(\underline{g}')\varphi^5, \tag{4.5}$$

with φ smooth and strictly positive, and where $\underline{g}' = \varphi^4 \underline{g}$ and $R(\underline{g}') = I(\underline{g})$. The study of this problem has attracted much effort (see [12] for a nice account). The difficulty in dealing with (4.5) lies in the well-known fact that the exponent of φ in the non-linear term in (4.5) is just the limiting exponent for the validity of the compactness of the Sobolev embedding theorem, so that we cannot use with confidence the variational method for solving (4.5). There are also other subtlities in (4.5) which are spelled in details in Aubin's book [12]. In any case, owing to the efforts of Yamabe, Trudiger, and Aubin, we now know that

$$I(\underline{g}) \leqslant 6\omega^{2/3}$$

for any riemannian structure $(S, \underline{g})$, normalized to unit volume, on a closed three-manifold S. Here ω is the volume of the unit three-sphere $\mathbb{S}^3$ endowed with its constant curvature metric $\underline{g}_{\text{can}}$. If furthermore

$$(4.6) \qquad I(\underline{g}) < 6\omega^{2/3},$$

then there exists a strictly positive solution of problem (4.5), with $R(\underline{g}' = \varphi^4 \underline{g}) = I(\underline{g})$ and $\int \varphi^6 \mathrm{d}v_{\underline{g}} = 1$, so that in this case the Yamabe conjecture holds true. [Notice that the three-dimensional Yamabe conjecture would be completely settled in the affermative if $I(\underline{g}) = 6\omega^{2/3}$ were to imply that $(S, \underline{g}) = (S^3, \underline{g}_{\text{can}})$].

We have seen that $I(\underline{g})$ carries information on the conformal structure associated with the model physical space $(S, \underline{g})$. However, what is much more interesting is the fact that $I(\underline{g})$ carries also information on the topology of the underlying manifold S. This easily follows by observing that conformally flat structures on topologically different base manifolds, say as $(\mathbb{S}^3, \underline{g}_{\text{can}})$ and $(T^3, \underline{g}_{\text{flat}})$, (the three-torus with the flat metric), gives rise to different values of $I(\underline{g})$ ($6\omega^{2/3}$ and 0, respectively). This latter remark suggests that a consistent way of obtaining topological information from the Hamiltonian constraint may be that of looking at the properties of the $I(\underline{g})$ associated with the given initial data set. What is particularly relevant is the sign of $I(\underline{g})$. A negative value for $I(\underline{g})$ does not appear to be interesting, since, as already recalled, there are no topological obstructions to negative scalar curvature on closed manifolds (actually it turns out that every three-dimensional manifold S admits conformal structures with $I(\underline{g}) < 0$ [35]). On the other hand, positive (more in general non-negative) scalar curvature is topologically obstructed, thus those collections of initial data sets such that $I(\underline{g}) > 0$ ($\underline{g}$ the physical metric) must be necessarily be supported by three-manifolds S compatible with such obstructions. In fact, as a consequence of Aubin's proof of the Yamabe conjecture, a positive $I(\underline{g})$ implies the existence, in the same conformal equivalence class $\{\underline{g}\}$ associated with $\underline{g}$ (hence within the same York data set), of a representative element $\underline{g}'$ such that $R(\underline{g}') > 0$ (actually $R(\underline{g}') = I(\underline{g})$).

The above-mentioned obstructions to positive scalar curvature on a riemannian manifold $(V^n, \underline{g})$ ($n \geqslant 3$) first made their appearance in a work by Lichnerowicz [38], who via the index theorem of Atiyah and Singer showed that if a spin-manifold admits a metric of non-negative scalar curvature (not identically zero) then its Hirzebuch $\hat{A}$-genus must be zero. This result was further extended by Hitchin [39], and by Gromov and Lawson [40], [41], who by using the Dirac operator (again on spin-manifold) generalized the Lichnerowicz argument. Obstructions to positive scalar curvature not costrained to spin manifold, were also investigated, by different, methods, by Schoen and Yau [8], who by examining the obstructions to the existence of stable minimal hypersurfaces in $(S, \underline{g})$, have been able to classify those, closed, simply connected orientable three-manifolds S which can support Riemannian structures of positive scalar curvature. Schoen and Yau's approach is particularly simple and fits well within the conformal method. In connecting the nature of the minimal surfaces in $(S, \underline{g})$ with the sign of the scalar curvature $R(\underline{g})$, it ultimately relates the structure of possible apparent horizons in the physical space with a geometrical object, $R(\underline{g})$, that via the Hamiltonian constraint, reflects the structure of the physical sources. [Strictly speaking, minimal two-surfaces are distinct from apparent horizons for non-time-symmetric data, (see p. 25), however it is a standard conjecture that whenever a minimal two-surface forms then, in the same isotopy class, an apparent horizon develops].

Schoen and Yau's method is essentially based on the second variation formula for the hypersurface area (3.25). From (3.25) and (3.28), assuming the minimality and the stability of the embedding of a closed two-surface $i(\partial \Sigma)$ in a three-manifold of positive curvature $(S, \underline{g})$, we get

$$\int_{i(\partial \Sigma)} |{}^{(2)}\nabla f|^2 + \frac{1}{2} (H - |\underline{b}|^2 - R(\underline{g})) \, d\sigma_{\underline{h}} \geqslant 0,$$

for any smooth function on $i(\partial \Sigma)$. For $f = 1$, this yields

$$\int_{i(\partial \Sigma)} H d\sigma_{\underline{h}} \geqslant \int_{i(\partial \Sigma)} (|b|^2 + R(\underline{g})) \, d\sigma_{\underline{h}} > 0$$

which, by the Gauss-Bonnet theorem, implies

(4.7) $\chi(\partial \Sigma) > 0,$

where $\chi(\partial \Sigma)$ is the Euler characteristic associated with $(\partial \Sigma)$. Thus according to (4.7), if $(S, \underline{g})$ has strictly positive scalar curvature and if it contains closed stable minimal surfaces such a $i(\partial \Sigma)$, then necessarily these surfaces must have positive Euler characteristic. In particular it follows from this result that a $(S, \underline{g})$ with $R(g) > 0$ cannot have closed immersed stable minimal two-manifolds the genus of which $\mathscr{G}$ is $\geqslant 1$, (as is known, the genus of a two-manifold, commonly the «number of handles» of the manifold is related to the Euler characteristic via $\chi(\partial \Sigma) = 2 - 2\mathscr{G}$). However, Schoen and Yau also showed that whenever the fundamental group $\Pi_1(S)$ of S has a subgroup which is isomorphic

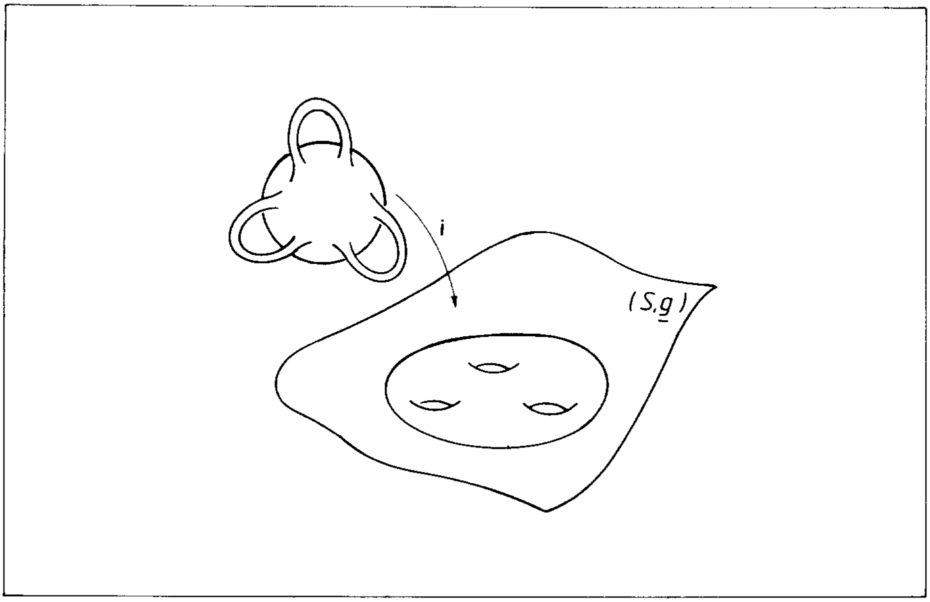

Figure 7. An immersed minimal 2-manifold of genus 3. If this minimal surface is stable then the scalar curvature of the ambient manifold (S, g) cannot be positive.

to $\Pi_1(\partial\Sigma_{\mathcal{G}})$ ($\partial\Sigma_{\mathcal{G}}$ being the standard two-sphere with $\mathcal{G}$, $\mathcal{G} \geq 1$, handles) there exists a stable minimal immersion of $\partial\Sigma_{\mathcal{G}}$ in S. It is thus clear, according to the previous results, that any such three-manifold S cannot support positive scalar curvature three-metrics. Hence for a closed three-manifold to accomodate a metric with positive scalar curvature, the fundamental group $\Pi_1(S)$ cannot contain a subgroup which is isomorphic to the fundamental group of a closed two-surface of genus ≥ 1. This fact provides a strong constraint for such S, and modulo some standard conjectures in the topology of three-manifolds, it shows that any closed three-manifold with positive scalar curvature is either the three-sphere $\mathbb{S}^3$ (possibly quotiented by a finite group of isometries acting without fixed points), or the handled three-sphere $S^2 \times S^1$ (this is a standard wormhole model for a closed three-manifold; there are some Gowdy (see [9] for references) spacetimes which exhibit this topology in their space sections), or a connected sum of a countable number of such three-manifolds, that is

$$S \simeq S^3/\Gamma \# \ldots \# S^3/\Gamma' \# S^2 \times S^1 \# \ldots \# S^2 \times S^1, \tag{4.8}$$

where $\Gamma, \Gamma', \ldots$, are finite groups in $SO(4)$.

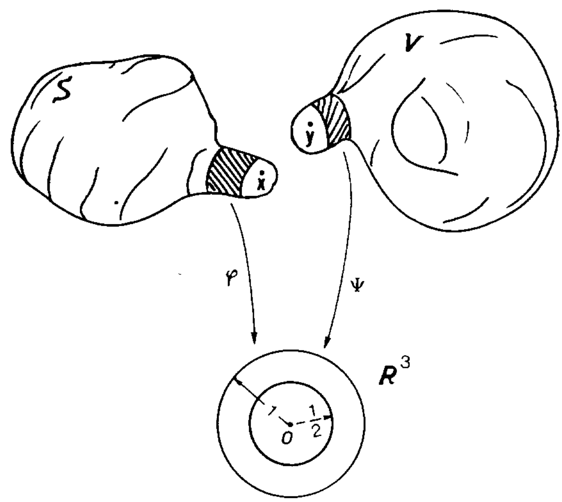

Figure 8. The definition of the connected sum of two manifold. Let φ and Ψ be local charts mapping a neighborhood of $x \in S$ and of $y \in V$ on $B(0,1) \in \mathbb{R}^n$ (the ball of radius 1 centered at the origin) with $\varphi(x) = \Psi(y) = 0$. Let h denote the inversion, in $\mathbb{R}^n$, of center 0, which preserves the collar $C(0, 1/2, 1)$ and which exchanges $S(0, 1/2)$ and $S(0,1)$ (the spheres of radius 1/2 and 1, centered in 0, respectively). Then

$$S \# V \equiv (S - \varphi^{-1}(B(0,1/2))) \underset{\Psi^{-1}\circ h \circ \varphi}{\cup} (V - \Psi^{-1}(B(0,1/2))$$

is the manifold obtained by glueing the manifolds $S - \varphi^{-1}(B(0,1/2))$ and $V - \Psi^{-1}(B(0,1/2))$ along the open sets $\varphi^{-1}(C(0,1/2,1)$ and $\Psi^{-1}(C(0,1/2,1)$ by means of the mapping $\psi^{-1} \circ h \circ \varphi$.

According to these results we are naturally led to discuss for what initial data sets the conformal invariant $I(\underline{g})$ is positive, for, such data can exist on S only if S is of the topological type (4.8). From the definitions (4.2) and (4.3) it immediately follows that $I(\underline{g}) > 0$ whenever $R(\underline{g}) > 0$. However, as already stressed, we would like to weaken this condition, for in realistic situation the scalar curvature of the physical space $(S, \underline{g})$ will be somewhere positive and somewhere else negative. In any case, an overall negative scalar curvature, regardless of how small, cannot be associated with a riemannian structure $\underline{g}$ satisfying $I(\underline{g}) > 0$. To show this, let us suppose, on the contrary, that such a $\underline{g}$ with $R(\underline{g}) < 0$ exists. by taking into account the variational definition (4.3) of $I(\underline{g})$ and evaluating the functional $Y(\underline{g}, f)$ for $f = 1$, we get

$$(4.9) \qquad (\mathrm{Vol}\,(S,\underline{g}))^{-1/3}\int_S R(\underline{g})\,\mathrm{d}v_{\underline{g}} \geqslant I(\underline{g}).$$

Hence, if $I(\underline{g}) > 0$, then we must necessarily have

$$(4.10) \qquad \int_S R(\underline{g})\,\mathrm{d}v_{\underline{g}} \geqslant 0.$$

Thus, according to (4.9) and (4.10), we expect that $I(\underline{g}) > 0$ is associated either with a riemannian structure the scalar curvature of which is everywhere positive, or with a riemannian structure with scalar curvature somewhere positive and somewhere negative, but on the average positive. In effect it is simple to prove that this positivity on the average is also a sufficient condition for having $I(\underline{g}) > 0$. As an easy computation shows [see p. 16, or p. 126 of [12]; it is this computation that implies the conformal invariance of $I(\underline{g})$], we have

$$(4.11) \qquad Y(\underline{g}, f) = Y(\underline{g}^*, uf)$$

whenever $\underline{g} = u^4\underline{g}^*$. For $f = 1$, on taking into account the definition (4.2) of the functional Y, (4.11) yields

$$(4.12) \qquad \begin{aligned} (\mathrm{Vol}\,(S,g))^{-1/3}\int_S R(\underline{g})\,\mathrm{d}v_{\underline{g}} &= \left(\int_S u^6\,\mathrm{d}v_{\underline{g}^*}\right)^{-1/3} \times \\ &\times \left(8\int_S (|\nabla^* u|^2 + R(\underline{g}^*)u^2)\,\mathrm{d}v_{\underline{g}^*}\right). \end{aligned}$$

Now let us assume that even if $\int R(\underline{g})\,\mathrm{d}v_{\underline{g}} > 0$, $I(\underline{g}) = I(\underline{g}^*) < 0$. This latter hypothesis, by the Aubin-Yamabe theorem, implies the existence of a smooth and strictly positive φ such that

$$-8\Delta^*\varphi + R(\underline{g}^*)\varphi^5 = I(\underline{g}^*)\varphi^5.$$

Substituting for $u = \varphi$ into (4.12) yields

$$(\mathrm{Vol}\,(S,\underline{g}))^{-1/3}\int_S R(\underline{g})\,\mathrm{d}v_{\underline{g}} = I(\underline{g}) < 0,$$

and hence a contradiction if, as assumed, $R(\underline{g})$ is on the average potive. Thus $I(\underline{g}) > 0$ whenever $\int R(\underline{g})\,\mathrm{d}v_{\underline{g}} > 0$.

Now, let us suppose that the metric $\underline{g}$ is the physical metric and let us take into account the Hamiltonian constraint in the form (4.1). Then the above results show that the topology of the three-manifold S modelling the physical space is necessarily that of a three-sphere, or that of a $(S^1 \times S^2)$-wormhole, or a connected sum thereof (see (4.8)), not only when on S there is enough matter and radiation such that

$$(4.13) \qquad 16\pi\mu + \underline{\tilde{K}} \circ \underline{\tilde{K}} - \frac{2}{3}(k)^2 > 0$$

holds pointwise, but, more in general, also when (4.13) is locally violated, provided that the negative contribution of $-\frac{2}{3}(k)^2$ is dominated on the average by by the matter plus radiation term $16\pi\mu + \underline{\tilde{K}} \circ \underline{\tilde{K}}$, that is when

$$(4.14) \qquad \int_S \left(16\pi\mu + \underline{\tilde{K}} \circ \underline{\tilde{K}} - \frac{2}{3}(k)^2\right) dv_{\underline{g}} > 0.$$

This relation is clearly not a condition to be imposed on the initial data, for in it there appear the physical data rather than the corresponding York data. In this sense, it is not a condition that may be used to select, a priori, the topology of S; rather, it may be used to explain, in a precise way, how the presence of matter and radiation works towards selecting a particular class of topologies for a closed physical space.

The above considerations suggest to define as generic the underlying topology of a closed physical space $(S, \underline{g})$ if there are no a priori restrictions on the sign of the conformal invariant $I(\underline{g})$. Notice that, as already recalled, on every closed three-manifold S a riemannian structure $\underline{g}'$ can be introduced such that $I(\underline{g}') < 0$. Furthermore [15], it can be shown that if S admits a riemannian structure the associated scalar curvature of which is non-negative, then S can support also a riemannian metric $\underline{g}$ with $I(\underline{g}) = 0$. According to such remarks, the only three--manifolds S modelling a closed physical space, the underlying topology of which is generic in the above sense, are either $\mathbb{S}^3$, $\mathbb{S}^3/\Gamma$, $S^1 \times S^2$, or a connected sum thereof.

The importance of the notion so introduced lies in the fact that, within the context of the conformal approach to the initial value problem, the genericity of S implies the existence of no a priori restriction on the possible free initial data S can support. For, $I(\underline{g})$ being a conformal invariant, an obstruction to particular values of the sign of $I(\underline{g})$ corresponds to further constraints on the possible initial data sets besides the for differential constraints (2.5), (2.6) naturally associated with the field equations. Clearly, as far as the field equations are concerned, the a priori existence of further constraints, besides the natural ones, must be considered as associated with a gravitational configuration not corresponding to the most general gravitational configuration admissible on a closed physical space. In this sense, other topologies for S than $\mathbb{S}^3$, $\mathbb{S}^3/\Gamma$, $S^1 \times S^2$, or a connected sum thereof unnaturally restrict the types of conformal structures S can support. For instance, on the three-torus T^3 there cannot be distribution of initial data with $I(\underline{g}) > 0$. In practice, genericity in the sense introduced above corresponds, via the Aubin-Yamabe theorem, to the freedom in choosing an arbitrary scale geometry on S (i.e. any $R(\hat{\underline{g}})$ in the Lichnerowicz-York equation (2.8)). The fact that some topologies leave less room than other topologies to possible initial data sets does not surprise. The case of the three-torus T^3, already recalled is typical. Here the obstruction to $I(\underline{g}) > 0$ corresponds to the impossibility of having initial data on T^3 such that $R(\underline{g}) = 16\pi\mu + \underline{K}\,\underline{K} - \frac{2}{3}(k)^2 > 0$. In particular, it immediately comes out from such remark that a spacetime $(V^4, {}^{(4)}\underline{g})$ development of some regular initial data set supported on $S \simeq T^3$, cannot admit a moment of time symmetry (i.e. a t_0 such that $\underline{K}_{t_0} = \underline{0}$), or, in general, a moment of maximum expansion (i.e. a t_0 such that $(k_{t_0}) = 0$), unless, at that instant, no sources are present- a well known circumstance that our results extend to all spacetimes development of regular initial data sets supported on space sections the topology of which differ from (4.8).

It is perhaps surprising that the amount of topological information that the Hamiltonian constraint can provide is so large. As we recalled, by itself, the problem of existence, uniqueness and statibility of solutions of the Lichnerowicz-York equation (2.8) does not involve much of the topology of S, when S is closed. It is rather the circumstance that the initial data (in particular, the gravitational initial data) must be specified in conformal equivalence classes that allows us to obtain, by means of the conformal invariant $I(\underline{g})$, topological information on S. In this way we argued that the topologies that we intuitively expect to be encountered in physically plausible world models are those actually preferred, either because the presence of matter and radiation favours them, or because they are less restrictive as arenas for possible initial data sets. It is difficult not to see in these results a further indication towards interpreting the action of the constraints, for a closed physical space, as a manifestation of Mach's principle [42].

5. SOME CONCLUDING REMARKS: TOPOLOGY OF THE PHYSICAL SPACE AND QUANTUM FLUCTUATIONS

Although the results of the previous paragraph show that there exists a deep connection between the topology of the physical space and its energetic content, we get somehow discouraged when we realize that at a classical level topology changes are forbidden. Wheeler suggested the idea that this is no more the case at the quantum level and that the topology of three-space might undergo quantum fluctuations when the radius of curvature in (S, g) becomes comparable with the Planck length. As DeWitt [1] has recently emphasized there are a number of conceptual difficulties that this picture must solve before being considered realistic. The most serious difficulty seems to be the suggestion, put forward in Anderson and DeWitt's paper, that topological changes are, as in the classical case, dynamically forbidden. This conclusion comes from examining an heuristic model whereby we replace the gravitational field with a linear massless scalar field. This field is supposed to evolve in a $(1 + 1)$-(space + time)-dimensional spacetime, where the space sections, originally diffeomerphic to the circle S^1, splits, as the evolution goes on, into two circles, i.e. $S^1 \to S^1 + S^1$. A discussion of this model, as well as of other, more refined, higher dimensional models, shows that a change in topology is always accompained by an infinite energy production. This fact comes out to be in conflict with the maintenance of the constraints before and after the topology change and with the non-violation of the usual causality requirements. Hence it forces the actual suppression of any change in the topology of the physical space.

However, as we have seen, the development of some sort of singularity as the topology of the physical space changes is not the whole story as far as the field equations are concerned. For, apparent horizons, disconnecting the singular regions from observations, do form «soonafter» the appearance of singularieis. This suggests an alternative view whereby the maintenance of the constraints is preserved «almost» always; topology changes, and quantum topology fluctuations are allowed, and give rise to singularities which soon after their appearance are hidden by apparent horizons. The correctness of this view is only very weakly supported by our considerations, and most likely it suffers by more serious conceptual difficulties than the standard view, however it remains a suggestion worth to be tested.

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